Final Exam – Linear Algebra and Image Processing (I and E)

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 10 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

Find the parametric vector form of the general solution of the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 5 & 0 & 0 & -2 & 6 \\ 2 & 10 & -2 & 1 & -3 & 16 \\ 0 & 0 & 2 & -2 & -4 & -8 \\ -1 & -5 & 2 & -2 & -2 & -17 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 3 \end{bmatrix}.$$

Question 2

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true).

(a) Every homogeneous linear system is consistent.

(b) Every linear map $\mathbb{R}^3 \to \mathbb{R}^2$ is onto (surjective).

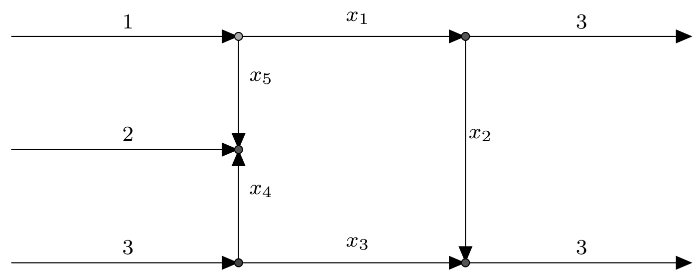
(c) Given an $n \times n$ matrix A, let B denote the matrix obtained from A by interchanging two rows. Then $\det A = \det B$.

(d) For any two $n \times n$ matrices A and B, we have that $(AB)^T = A^T B^T$ (where M^T denotes the transpose of M).

(e) A linear map $\mathbb{R}^3 \to \mathbb{R}^3$ is onto (surjective) if and only if it is one-to-one (injective).

Question 3

Consider the following network:



(a) Write down a linear system describing the flow in the network.

(b) Put the augmented matrix of the linear system from (a) in row reduced echelon form.

(c) Write the general solution of the linear system in parametric vector form.

Question 4

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}, \quad , B = \begin{bmatrix} k & 1 \\ 0 & 2 \end{bmatrix}.$$

(a) For which real numbers k is $A + B = B + A$?

(b) For which real numbers k is $AB = BA$?

(c) Find a non-zero 2×2 matrix C such that AC is the 2×2 zero matrix.

Question 5

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 3 & 2 & 1 \end{bmatrix}, \quad , B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 6 & 5 & 4 & 0 & 0 \\ 7 & 5 & 3 & 1 & 1 \\ 5 & -2 & -3 & -4 & 5 \end{bmatrix}.$$

(a) Compute the determinant $\det A$.

(b) Compute the determinant $\det B$.

(c) Compute the determinant $\det(A^3)$.

Question 6

Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 - x_3 \\ -3x_1 + x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix}.$$

Write $V = \operatorname{im} T$ for the image of T, and $W = \ker T$ for the kernel of T.

(a) Write down the standard matrix of T.

(b) Compute the dimension of V.

(c) Compute the dimension of W.

[Hint: Note that $\dim V + \dim W = \dim \mathbb{R}^3$].

Question 7

Exactly one of the following sets of vectors in $\mathbb{R}^4$ is linearly independent:

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 17 \\ 2 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(a) Which of the sets S_1, S_2 is linearly independent?

(b) For the set which is linearly independent, expand it to a basis of $\mathbb{R}^4$.

Question 8

Let A be the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

(a) Show that $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue?

(b) Calculate all eigenvalues of A.

(c) Give a basis for each eigenspace of A.

(d) Is A diagonalisable? If so, give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not.

Question 9

Consider the matrix A and the vector $\mathbf{b}$ given by

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -3 \\ -2 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 36 \\ 2 \end{bmatrix}.$$

(a) Show that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(b) Find the QR-decomposition of A.

(c) How many least-squares solutions are there?

(d) Use your answer from (b) to determine a least-squares solution.

Question 10

Let $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

(a) Show that the matrix $A^T A$ has eigenvalues 8 and 2.

(b) For each of the eigenvalues from (a), give an eigenvector of length 1.

(c) Give a singular value decomposition of A.