Final Exam – Lineaire Algebra 2

17 June 2016

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 6 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

Question 1

To answer this question, you will need the extra sheet. Please write your answers to this question on that sheet (you can use the back if you need it). Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A (so that the standard matrix of T is A). Draw the image of the rectangle in figure (a) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix B (so that the standard matrix of T is B). Draw the image of the triangle in figure (b) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

c) Write a 2×2 matrix C so that the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix T sends the triangle in figure (c) to the triangle in figure (d). You should write your answer on the extra sheet.

Question 2

For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.

a) The function $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ sending every vector in $\mathbb{R}^2$ to the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is linear.

b) Given an 2×2 matrix A, let B denote the matrix obtained from A by interchanging two rows. It is always true that $\det A = \det B$.

c) For every two 2×2 matrices A and B, we have that $\det(A - B) = \det(A) - \det(B)$.

d) Every homogeneous linear system is consistent (that is, has at least one solution).

e) For every pair A, B of $n \times n$, we have that $(AB)^T = A^T B^T$ (where M^T denotes the transpose of M).

Question 3

Define a matrix A by

$$A = \begin{bmatrix} -2 & -8 \\ 0 & -4 \end{bmatrix}.$$

a) Show that the eigenvalues of A are -2 and -4.

b) For each eigenvalue in (a) give a basis for the corresponding eigenspace.

c) Is A diagonalisable? If so, give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why not.

d) Calculate A^3.

Question 4

Let the matrix A and the vector $\underline{b}$ be given by

$$A = \begin{bmatrix} -1 & -2 & 3 \\ -1 & 0 & -3 \\ 1 & 0 & -3 \\ 1 & 2 & -3 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} 2 \\ 2 \\ 6 \\ 1 \end{bmatrix}.$$

a) Show that the system $A\underline{x} = \underline{b}$ is inconsistent (has no solutions).

b) Use the Gram-Schmidt process to turn the columns of A into an orthogonal set.

Question 5

Consider the following three vectors:

$$\underline{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ -3 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \underline{w} = \begin{bmatrix} 0 \\ 22 \\ 3 \\ 6 \\ 0 \\ -1 \end{bmatrix}.$$

a) Show that $\underline{u}$ and $\underline{v}$ are orthogonal.

b) Calculate the distance between $\underline{u}$ and $\underline{v}$.

c) Write S for the span of $\underline{u}$ and $\underline{v}$. Find a vector $\underline{\hat{w}}$ in S and $\underline{z}$ in the orthogonal complement $S^{\perp}$ such that $\underline{w} = \underline{\hat{w}} + \underline{z}$ (this $\underline{\hat{w}}$ is the orthogonal projection of $\underline{w}$ on S).

Question 6

1. What does it mean for a set of vectors in $\mathbb{R}^n$ to be *orthogonal*?

2. What does it mean for a set of vectors in $\mathbb{R}^n$ to be *orthonormal*?

3. Recall that an $n \times n$ matrix is called orthogonal if its columns form an orthonormal set. As a function of n, how many $n \times n$ orthogonal matrices with integer entries are there? Hint: try working out the answer for $n = 1$, $n = 2$ first.