Universiteit Leiden
Wiskunde en Natuurwetenschappen

Vak: LA2- Retake

Naam: D. Holmes

Datum: 11 July 2017.

Studierichting:

Docent: D. Holmes

Collegekaartnummer:

1a) $\det\begin{pmatrix} 4-\lambda & 0 & 2 \\ 3 & 1-\lambda & 2 \\ -15 & 3 & -6-\lambda \end{pmatrix} = (4-\lambda)\begin{bmatrix} 1-\lambda & 2 \\ 3 & -6-\lambda \end{bmatrix} + 2\begin{bmatrix} 3 & 1-\lambda \\ -15 & 3 \end{bmatrix}$

$= (4-\lambda)(-6+6\lambda-\lambda+\lambda^2-6) + 2(9+15-15\lambda)$

$= (4-\lambda)(\lambda^2+5\lambda-12) + 48 - 30\lambda$

$= 4\lambda^2 + 20\lambda \cancel{-48} - \lambda^3 - 5\lambda^2 + 12\lambda \cancel{+48} - 30\lambda$

$= -\lambda^3 - \lambda^2 + 2\cancel{8}\lambda = -\lambda(\lambda^2 + \lambda \overset{-2}{\cancel{28}}) = -\lambda(\lambda+2)(\lambda-1)$

e-values 0, 1, -2.

b) $\lambda=0$: $\left(\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 3 & 1 & 2 & 0 \\ -15 & 3 & -6 & 0 \end{array}\right) \to \left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 4 & 0 & 2 & 0 \end{array}\right]$

$\to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_1 = -\frac{x_3}{2}$, $x_2 = -\frac{x_3}{2} \to \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ eigenvector

$\lambda=1$: $\left[\begin{array}{ccc|c} 3 & 0 & 2 & 0 \\ 3 & 0 & 2 & 0 \\ -15 & 3 & -7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 15 & -3 & 7 & 0 \end{array}\right]$

$\to \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & -3 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = -\frac{2x_3}{3}$

$x_2 = -x_3$ e-vector $\begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$

$\lambda=-2$: $\left[\begin{array}{ccc|c} 6 & 0 & 2 & 0 \\ 3 & 3 & 2 & 0 \\ -15 & 3 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ -3 & 3 & 0 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ e-vector $\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

$\lambda=0$: basis $\left\{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right\}$

$\lambda=1$: basis $\left\{\begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}\right\}$

$\lambda=-2$: basis $\left\{\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}\right\}$.

2) $\det\begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -2 \\ -1 & 1 & 3 \end{bmatrix} = -4 \cdot \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$

$= -4(3-2) - (-2-2) = -4-(-4) = 0.$

a) No, not lin. indep.

b) No. Can remove a vector without changing span, so dim Span S ≤ 2.

c) Know dim Span S ≤ 2. First two vectors clearly lin. indep., so dim Span S $= 2$.

3) a) $\underline{u} \cdot \underline{v} = 3-2-1=0$ Yes, orthogonal.

b) $\text{dist}(\underline{u}, \underline{v}) = \sqrt{\|\underline{u}-\underline{v}\|} = \sqrt{\left\| \begin{matrix} 2 \\ 3 \\ -2 \end{matrix} \right\|} = \sqrt{4+9+4}$

$= \sqrt{17}.$

c) $\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ $\quad \underline{w} \cdot \underline{u} = 0$, $\quad \underline{w} \cdot \underline{v} = 0$ $\quad$ now $\underline{w}$ is a sol'n of linear system

with ACM $\left[\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$.

Row reduce: $\left[\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -4 & 0 \end{array}\right]$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{4}{5} & 0 \end{array}\right]$$

$$w_1 = -\frac{w_3}{5}$$

$$w_2 = \frac{4w_3}{5}$$

So $\underline{w} \in \text{Span}\left\{\begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}\right\}$.

$$\text{len}\begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix} = \sqrt{1+16+25} = \sqrt{42}.$$

So $\underline{w} = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ (or $\underline{w} = \begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$).

4) a) $\det\begin{pmatrix} 17-\lambda & -12 \\ 24 & -17-\lambda \end{pmatrix} =$

$$-(17-\lambda)(17+\lambda) + 12\cdot 24$$

$$= -17^2 + \lambda^2 + 12\cdot 24$$

$$= \lambda^2 + 240 + 48 - 17^2$$

$17^2 = 170 + 70 + 49$

$= 240 + 49 = 289$

$$= \lambda^2 + 288 - 289 = \lambda^2 - 1.$$

$$= (\lambda+1)(\lambda-1)$$

Or find eigenvectors.

Universiteit Leiden
Wiskunde en Natuurwetenschappen

Vak: LA2 retake Naam: D. Holmes.

Datum: Studierichting:

Docent: Collegekaartnummer:

4b) $\lambda=1$: $\left(\begin{array}{cc|c} 16 & -12 & 0 \\ 24 & -18 & 0 \end{array}\right) \to \left[\begin{array}{ccc} 4 & -3 & 0 \\ 4 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right)$

basis $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$

$\lambda=-1$: $\left[\begin{array}{cc|c} 18 & -12 & 0 \\ 24 & -16 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$ basis $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$.

c) $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $P = \begin{pmatrix} -3 & -2 \\ 4 & 3 \end{pmatrix}$

d) $A^{100} = PD^{100}P^{-1}$ $D^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= P I_2 P^{-1} = PP^{-1} = I_2$

$A^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

5a) $\underline{u}_1 = \begin{bmatrix}0\\-5\\5\end{bmatrix}$ $\underline{u}_2 = \begin{bmatrix}2\\2\\-2\end{bmatrix}$ $\underline{u}_3 = \begin{bmatrix}-5\\3\\-5\end{bmatrix}$.

$\underline{v}_1 = \underline{u}_1$

$$\underline{v}_2 = \underline{u}_2 - \frac{\underline{u}_2 \underline{v}_1}{\underline{v}_1 \underline{v}_1} \underline{v}_1$$

$\underline{u}_2 \cdot \underline{v}_1 = -20$

$\underline{v}_1 \cdot \underline{v}_1 = 50$

$$\cancel{\underline{v}_2 = \begin{bmatrix}0\\-5\\5\end{bmatrix} + \frac{2}{5}}$$

$$\underline{v}_2 = \begin{bmatrix}2\\2\\-2\end{bmatrix} + \frac{2}{5}\begin{bmatrix}0\\-5\\5\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}.$$

$$\underline{v}_3 = \underline{u}_3 - \frac{\underline{u}_3 \underline{v}_1}{\underline{v}_1 \underline{v}_1} \underline{v}_1 - \frac{\underline{u}_3 \underline{v}_2}{\underline{v}_2 \underline{v}_2} \underline{v}_2$$

$\underline{v}_1 \underline{v}_1 = 50$, $\underline{u}_3 \cdot \underline{v}_1 = -40$

$\underline{v}_2 \cdot \underline{v}_2 = 4$ $\underline{u}_3 \cdot \underline{v}_2 = -10$.

$$\underline{v}_3 = \begin{bmatrix}-5\\3\\-5\end{bmatrix} + \frac{4}{5}\begin{bmatrix}0\\-5\\5\end{bmatrix} + \frac{5}{2}\begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}0\\-1\\-1\end{bmatrix}$$

So orthogonal set is

$$\left\{ \begin{bmatrix}0\\-5\\5\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\end{bmatrix} \right\}.$$

5 b) Yes. $\begin{pmatrix} \text{GS basis} \\ \text{Orthogonal} \end{pmatrix} \Rightarrow \begin{pmatrix} \text{GS basis} \\ \text{linearly indep} \end{pmatrix}$, so

~~dim Span~~ dim Col $A = 3$, so ~~dim in~~

cols of A are lin indep.

c) Yes. ~~row ran~~ dim Row A = dim Col A = 3,

so rows must be lin indep.

d) ~~rank A = column rank = 3.~~
rank A = dim Col $A = 3$.

6 a) False; scalar multiplication by any real number is linear.

b) True. Let $\underline{x}_1, \underline{x}_2 \in U \cap V$, & $\lambda \in \mathbb{R}$.

Then $\underline{x}_1 + \underline{x}_2 \in U$ & $\underline{x}_1 + \underline{x}_2 \in V$ so $\underline{x}_1 + \underline{x}_2 \in U \cap V$

and $\lambda \underline{x}_1 \in U$ & $\lambda \underline{x}_1 \in V$ so $\lambda \underline{x}_1 \in U \cap V$.

c) True. A, B invertible $\Rightarrow \det A \neq 0$ & $\det B \neq 0$.

So $\det(AB) = (\det A)(\det B) \neq 0$.

d) False, eg $n = 1$, $A = [1]$, $B = [-1]$.

e) False, there are two: $[5]$ & $[-5]$.