Final Exam – Lineaire Algebra 1

13 January 2017

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 6 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

1. Define a matrix A and vector $\underline{b}$ by

$$A = \begin{bmatrix} 2 & 3 & 3 & -3 \\ -4 & -4 & -2 & 2 \\ 3 & 0 & -3 & 3 \end{bmatrix}, \underline{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

   a) Write down the augmented coefficient matrix of the linear system $A\underline{x} = \underline{b}$.

   b) Row reduce the augmented coefficient matrix that you wrote down.

   c) Write down 2 distinct solutions of the linear system $A\underline{x} = \underline{b}$. Hint: first, re-write the augmented coefficient matrix as a system of linear equations.

2. Define a matrix A by

$$A = \begin{bmatrix} -1 & 1 & 5 \\ 5 & 0 & 4 \\ -1 & 0 & -1 \end{bmatrix}.$$

   a) Compute the determinant of A.

   b) Find the inverse of A.

   c) What is the determinant of A^{-1}?

3. To answer this question, you will need the extra sheet. Please write your answers to this question on that sheet (you can use the back if you need it). Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

   a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A (so that the standard matrix of T is A). Draw the image of the rectangle in figure (a) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

   b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix B (so that the standard matrix of T is B). Draw the image of the triangle in figure (b) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

c) Write a 2×2 matrix C so that the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix T sends the triangle in figure (c) to the triangle in figure (d). You should write your answer on the extra sheet.

4. Consider the matrix

$$L = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}.$$

a) Draw the graph (with 5 vertices) whose Laplacian matrix is the matrix L.

b) Let $L_{1,1}$ be the matrix obtained from L be deleting the first row and the first column. Compute the determinant of $L_{1,1}$.

c) How many spanning trees does the graph in (a) have?

d) Draw all of the spanning trees of the graph.

5. For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.

a) If A is a 3×4 matrix then the associated linear map must be injective.

b) If A and B are $n \times n$ matrices then $A^{-1} + B^{-1} = (A + B)^{-1}$.

c) If the last column of the augmented coefficient matrix of a linear system is a pivot column the the system is inconsistent.

d) Let A be an $n\times n$ matrix, and let B be the matrix obtained from A by multiplying every entry of A by 2. Then $\det B = 2n \det A$.

e) Let A be an $n \times n$ matrix with $n \geq q$. If A has two rows equal then $\det A = 0$.

6. Let $\underline{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose that

$$T\left(\underline{u}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } T\left(\underline{v}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

a) Compute $2\underline{v} - \underline{u}$.

b) Use the linearity of T to compute $T(2\underline{v} - \underline{u})$.

c) What is $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$?

d) Compute $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. Hint: try to write $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in terms of $\underline{u}$ and $\underline{v}$.

e) Write down the standard matrix for T.