Solutions to Final Exam – Lineaire Algebra 2

3 June 2019

Time: 3 hours.

1. a)

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) The solution set is the span of the vectors $\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

c) The dimension is 2.

2. a) Yes.

b) Yes.

c) No.

d) $\begin{bmatrix} \frac{13}{19} \\ \frac{65}{38} \\ -\frac{39}{38} \end{bmatrix}$.

3. a) False; almost any example with 2×2 matrices will suffice.

b) False; for example if $n = 3$, and we take A to be the identity matrix, then $\det B = 8 \neq 6$.

c) True; just check the axioms.

d) True, $(A - \lambda I_n)\underline{x} = \mathbf{0}$ is equivalent to $A\underline{x} = \lambda\underline{x}$.

e) True; the orthogonal complement to $\underline{a}$ has dimension 1, and so is spanned by $\underline{b}$, then use that $(V^{\perp})^{\perp} = V$.

4. a) Solution is not unique, but one basis of kernel given by $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

b) Basis of column space given by $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$.

c) The *image* is the set of vectors $\underline{v} \in \mathbb{R}^3$ such that there exists $\underline{u} \in \mathbb{R}^4$ with $T(\underline{u}) = \underline{v}$.

d) $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$.

5. a) For example, show that the given values are roots of the *characteristic polynomial* $\det(A - \lambda I_2) = \lambda^2 + \lambda - 6$.

b) A basis of the eigenspace is given by $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

c) A basis of the eigenspace is given by $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

d) $P = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$.

e) $A^{2019} = PD^{2019}P^{-1} = \begin{bmatrix} 3\,x^{2019} - 2\,y^{2019} & x^{2019} - y^{2019} \\ -6\,x^{2019} + 6\,y^{2019} & -2\,x^{2019} + 3\,y^{2019} \end{bmatrix}$ where $x = 2$ and $y = -3$. This simplifies to the given solution.

6. (a) $AA^T = \begin{bmatrix} 17 & -7 \\ -7 & 3 \end{bmatrix}$.

(b) $A^t\underline{b} = \begin{bmatrix} 15 & -7 \end{bmatrix}$.

(c) The ACM of the normal equation is $\begin{bmatrix} 17 & -7 & 15 \\ -7 & 3 & -7 \end{bmatrix}$. This row-reduces to $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -7 \end{bmatrix}$.

The unique solution is then given by $\begin{bmatrix} -2 \\ -7 \end{bmatrix}$.

(d) The best-fitting curve is obtained when $\underline{v} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$ is a least-squares solution to the equation

$$\begin{bmatrix} 1 & \sqrt{1} \\ 1 & \sqrt{1} \\ 1 & \sqrt{4} \\ 1 & \sqrt{4} \end{bmatrix} \underline{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}.$$

The $\underline{v}$ giving the best fit is then the solution to the normal equation

$$\begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix} \underline{v} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}.$$