Solutions to Final Exam – Lineaire Algebra 2
3 June 2019

Time: 3 hours.

1. a)

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) The solution set is the span of the vectors $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

c) No, as the solution set has dimension 2.

d) No, because any solution lies in the kernel.

2. a) $A^{-1} = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & -8 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\det B = -3$

c) $AB = \begin{bmatrix} 0 & 1 & -3 \\ -14 & -13 & -2 \\ -3 & -3 & 0 \end{bmatrix}$

d) $\det(-3B) = 81$

3. a) True; e.g. send $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$.

b) False; almost any example with $n = 2$ will do.

c) True; just write out the axioms;

d) True; the diagonal entries are the eigenvalues.

e) True; $(Ab) \cdot (Ab) = b^T A^T Ab = b^T A^{-1} Ab = b^T b = b \cdot b$.

4. a) Yes.

b) Yes.

c) No.

d) Projection is $\begin{bmatrix} \frac{2}{7} \\ -\frac{29}{14} \\ -\frac{9}{14} \end{bmatrix}$.

5. a) For example, show that the given values are roots of the *characteristic polynomial* $\det(A - \lambda I_2)$.

b) A basis of the eigenspace is given by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

c) A basis of the eigenspace is given by $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

d) $P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

e) $A^{2019} = \begin{bmatrix} 7 & -6 \\ 8 & -7 \end{bmatrix}$.

6. (a) We have $A\underline{y} = \underline{b}$, so the distance from $A\underline{y}$ to $\underline{b}$ is zero. Distances are always non-negative, so this evidently minimises the distance.

   (b) We know there exists a vector $\underline{y}$ which is an actual solution, so the distance from $A\underline{y}$ to $\underline{b}$ is 0. Thus any LSS $\underline{z}$ must also have that the distance from $A\underline{z}$ to $\underline{b}$ is zero, but the only vector of length 0 is the zero vector.

   (c) $AA^T = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$.

   (d) $A^t\underline{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

   (e) The ACM of the normal equation is $\left[\begin{array}{cc|c} 2 & -2 & 4 \\ -2 & 5 & 2 \end{array}\right]$. This row-reduces to $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right]$. The unique solution is then given by $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$.