LA1 Retake, 16 March 2018.

1a)
$$\begin{bmatrix} 2 & 2 & 1 & -4 \\ -2 & -2 & -4 & 4 \\ 1 & 1 & -4 & -2 \\ 3 & 3 & 2 & 1 \end{bmatrix} \xrightarrow[\text{then } r_2 \mapsto r_2 + 2r_3]{r_1 \mapsto r_1 + r_2} \begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 0 & -12 & 0 \\ 1 & 1 & -4 & -2 \\ 3 & 3 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 2 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) Equations:
$$\begin{aligned} x_1 + x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= -x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \\ x_4 &= 0. \end{aligned}$$

$$\text{Sol'n set} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

2a)
$$\det A = -\det \begin{bmatrix} 0 & -1 & -2 \\ 1 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix} = -\Big[1\cdot\big(1\times 0 - 1\times 1\big) - 2\big(1\times(-2) - 1\times(-1)\big)$$

$$= -\big[-1 - 2(-1)\big] = -1$$

$r_1 \mapsto -r_1$

$r_2 \leftrightarrow r_3$
$r_3 \mapsto -r_3$

2b) $$\left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$r_4 \mapsto r_4 - r_2$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow[r_4 \mapsto r_4 + r_3]{r_2 \mapsto r_2 + r_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \end{array}\right]$$

$r_2 \mapsto r_2 - 3r_4$
$r_3 \mapsto r_3 - 2r_4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 4 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 4 & -3 \\ 0 & 1 & 2 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

2c) $\det A^{-1} = \dfrac{1}{\det A} = \dfrac{1}{-1} = -1$

2d $\det 2A = 2^4 \det A = -16.$

3 a) $\frac{u}{2} - \frac{v}{2} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b) $T\left(\frac{u}{2} - \frac{v}{2}\right) = \frac{1}{2}T(u) - \frac{1}{2}T(v) = \frac{1}{2}\begin{bmatrix} 10 \\ 8 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$

c) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\frac{u}{2} - \frac{v}{2}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cancel{= (v) - \left(\frac{u}{2} - \frac{v}{2}\right)}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T(v) - T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

e) Standard matrix $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}.$

4a Linear: Say $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, Then $T(u+v) = M(u+v) = Mu + Mv$
$= T(u) + T(v)$
$T(cu) = M(cu) = c(Mu)$
$= cT(u).$

b) linear. $T(u+v) = 0 = T(u) + T(v)$
$T(cu) = 0 = cT(u).$

c) Not linear. $T\left[-1\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq -1\,T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

d) Linear: composite of linear functions is linear.

e) Linear because $f_b \circ f_c = f_b$ ~~Ab~~ which is linear.

5a) $x_1 + x_2 = 1$

$x_1 + x_3 = x_4$

$x_3 = x_5$

$x_4 = x_5 + 4$

$x_2 + 4 = 1$

b) $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 4 \\ 0 & 1 & 0 & 0 & 0 & -3 \end{array}\right] \xrightarrow{\text{swap rows}} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 4 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$r_1 \mapsto r_1 - r_3 + r_4$ then $r_5 \mapsto r_5 - r_1 - r_2$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

row-reduced echelon form.

c) x_5 is free, & $x_3 - x_5 = 0$, so by taking $x_5 > 0$ we find $x_3 > 0$.

6a) True. $\det\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

so $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible $\Rightarrow \det = 1 \Rightarrow \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$ invertible.

b) True. Let B be matrix obtained from A by dividing 1st row by 2, Then $\det A = 2 \det B$, & B has integer entries so $\det B$ is an integer.

c) False, eg $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\det A = 0$, even.

d) True, $\det(A^T) = \det A$, & a matrix is invertible $\Leftrightarrow$ determinant $\neq 0$

e) True. Colums span $\mathbb{R}^n \Rightarrow$ pivot in every row $\Rightarrow n$ pivots $\Rightarrow$ pivot in every column $\Rightarrow$ colums linearly independent.