Final Exam – Lineaire Algebra 2

19 June 2018

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 6 questions, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

1. Define a matrix A and a vector $\underline{b}$ by

$$A = \begin{bmatrix} -3 & 4 & 1 & 1 \\ 4 & -5 & -1 & 1 \\ 2 & 1 & 6 & -5 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}.$$

   a) Which of the following matrices is the result of applying the row-reduction algorithm to the augmented coefficient matrix of the equation $A\underline{x} = \underline{b}$?

$$M = \begin{bmatrix} 1 & 0 & 0 & 18 & -17 \\ 0 & 1 & 0 & 32 & 15 \\ 0 & 0 & 1 & -10 & 11 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 & 0 & 19 & -24 \\ 0 & 1 & 0 & 17 & -21 \\ 0 & 0 & 1 & -10 & 11 \end{bmatrix}.$$

   You should answer either M or N, and justify your answer.

   b) Write the general solution to the equation $A\underline{x} = \underline{b}$ in parametric vector form.

2. We define several vectors in $\mathbb{R}^3$:

$$\underline{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \quad \underline{d} = \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix}.$$

   a) Is the set $\{\underline{a}, \underline{c}\}$ orthogonal?

   b) Is the set $\{\underline{a}, \underline{b}\}$ orthogonal?

   c) Is the set $\{\underline{a}, \underline{b}, \underline{c}\}$ orthogonal?

   d) Compute the projection of the vector $\underline{d}$ onto the span of $\{\underline{a}, \underline{b}\}$.

3. For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.

   a) If all the entries of a 2×2 matrix are positive (> 0), then the determinant of the matrix is positive (> 0).

   b) If a 3×3 matrix A is diagonalisable then A must have three distinct eigenvectors.

   c) If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are non-zero vectors in $\mathbb{R}^2$ with $\underline{a}$ orthogonal to $\underline{b}$ and $\underline{b}$ orthogonal to $\underline{c}$, then it must hold that $\underline{a}$ is a scalar multiple of $\underline{c}$.

   d) If $\underline{u}$ and $\underline{v}$ are vectors in $\mathbb{R}^2$ then we must have $\underline{u} \cdot \underline{v} \leq \|\underline{u}\| \, \|\underline{v}\|$.

   e) If $\underline{u}$ and $\underline{v}$ are vectors in $\mathbb{R}^2$ then we must have $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$. *Hint: try writing out* $(\|\underline{u}\| + \|\underline{v}\|)^2 - \|\underline{u} + \underline{v}\|^2$, *then applying your answer to (d).*

4. We define matrices A and B by

$$A = \begin{bmatrix} 4 & 5 & 6 & -1 \\ -2 & -2 & -6 & 6 \\ 4 & 5 & 6 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 9 & -14 \\ 0 & 1 & -6 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the remainder of this question you may use without proof that A and B are row equivalent.

a) Give a basis for the null space $\mathrm{Null}(A)$.

b) Give a basis for the column space $\mathrm{Col}(A)$.

Now let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by $T(\underline{x}) = A\underline{x}$.

c) State the definition of the *kernel* of a linear transformation.

d) What is the dimension of the kernel $\ker(T)$?

5. We define a matrix A and vectors $\underline{u}$ and $\underline{v}$ by

$$A = \begin{bmatrix} -12 & -12 & 0 \\ 6 & 5 & 0 \\ -22 & -28 & 1 \end{bmatrix}, \quad \underline{u} = \begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}.$$

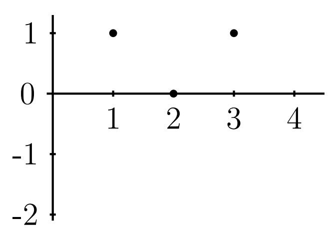
You may use without proof that -3 is an eigenvalue of A with eigenvector $\underline{u}$, and -4 is an eigenvalue of A with eigenvector $\underline{v}$.

a) A has exactly one other eigenvalue. What is it?

b) Find an eigenvector for the eigenvalue you found in (a). *Check that it is actually an eigenvector.*

c) Write a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

6. An experiment gave the following datapoints: $(1, 1), (2, 0), (3, 1)$, see the graph below.



(a) Use the 'normal equation' $A^T A = A^T \mathbf{b}$ to find the least squares line that best fits these points.

(b) Show that a curve of the form $y = a + bx + cx^2$ that best fits these points is given by $y = 4 - 4x + x^2$. *Hint: it is enough to check that $y = 4 - 4x + x^2$ corresponds to a solution to a suitable normal equation. You should not need to carry out any row reduction, though you can if you want.*