Rest. "Pakhuis"; 18^{45}, Doelensteeg 8

Completely bounded maps and operator algebras.

Typical operator algebra is a closed subalgebra of $B(H)$

C^*-algebras:

Def. A Banach algebra with involution such that $\|a^* a\| = \|a\|^2$, is called C^*-algebra

Thm. (GNS) Let A be a C^*-algebra, then there exists a Hilbert space H and an isometric $*$-homomorphism $\pi: A \to B(H)$

The GNS theorem is proved using states. This requires the notion positivity in a C^*-algebra.

$$A_+ := \{a^* a \mid a \in A\}$$
$$= \{a \mid a = a^*, \ \sigma(a) \subset \mathbb{R}_{\geq 0}\}$$

The set A_+ is a closed convex cone. In fact $A = \mathrm{span}_{\mathbb{C}}(A_+)$.

Def. A linear map $\varphi: A \to B$ is positive if $\varphi(A_+) \subset B_+$

$a \geq 0 \Rightarrow \varphi(a) \geq 0$

A state is a positive linear functional $\phi: A \to \mathbb{C}$ of norm 1. Given a state

$$\langle \, , \, \rangle_\phi : A \times A \to \mathbb{C}$$
$$(a,b) \mapsto \langle a, b \rangle_\phi := \phi(a^* b)$$

$$H_\phi := \overline{A}^{\langle \, , \, \rangle_\phi}$$ and a $*$-rep $\pi: A \to B(H_\phi)$.

A C^*-algebra has "enough" states, $\forall a \in A \ \exists$ state ϕ $|\phi(a)| = \|a\|$

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A **state** is a positive linear functional $\phi : A \to \mathbb{C}$ of norm 1. Given a state
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A C^*-algebra has "enough" states, $\forall a \in A \ \exists$ state ϕ $|\phi(a)| = \|a\|$

$H_{GNS} = \bigoplus_{\phi \text{ states}} H_\phi$

Gelfand-Naimark Theorem for commutative C^*-algebras:
If A is a commutative C^*-algebra then there exists a locally compact Hausdorff space X and a $*$-isomorphism $\pi : A \xrightarrow{\sim} C_0(X)$ $\left(\|f\| := \sup_{x \in X} |f(x)|, \ f^*(x) = \overline{f(x)}\right)$.

Continuous functional calculus:
Let $a \in A$ be a normal element, $a^*a = aa^*$
Then $C^*(a) \simeq C_0(\sigma(a))$

The isomorphism $C^*(a) \simeq C_0(\sigma(a))$ maps a to $f(t) = t$.
Fact $\sigma(f(a)) = \{ f(\lambda) \mid \lambda \in \sigma(a) \}$.
Using the functions
$$f^+(x) = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$
$$f^-(x) = \begin{cases} 0 & x \geq 0 \\ -x & x \leq 0 \end{cases}$$
We have for $a = a^*$ $a = f^+(a) - f^-(a)$
$= h + k$

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For general $a \in A$:

$a = \frac{1}{2}(a + a^*) + \frac{i}{2}(ia^* - ia)$

$= h + ik$, $h^* = h$, $k = k^*$

$= h^+ - h^- + ik^+ - ik^-$

Matrix norms:

If A is a C^*-algebra, then so is $M_n(A)$. Choose $\pi: A \to B(H)$, then since $B(H^n) \simeq M_n(B(H))$

$\pi_n: M_n(A) \to M_n(B(H))$

$(a_{ij}) \mapsto (\pi(a_{ij}))$

makes $M_n(A)$ into a C^*-algebra.

(Alternative method: Equip A^n with the "right" norm

$\left\|\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right\|^2 := \left\| \left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right\rangle \right\| := \left\| \sum_{i=1}^n a_i^* a_i \right\| \left(\leq \sum_{i=1}^n \|a_i^* a_i\| \right)$

Hilbert C^*-modules.

$M_n(A) = M_n(\mathbb{C}) \otimes A$.

All this works for $S \subset A$ closed subspace!

For $S \subset A$ closed linear subspace $M_n(S)$ carries a canonical norm $\|\cdot\|_n$

Intrinsic definition:

Let S be a vector space with for each n a norm on $\|\cdot\|_n$ on $M_n(S)$, such that:

- $M_n(S)$ is Banach for $\|\cdot\|_n$.
- for all $A, B \in M_n(\mathbb{C})$ and $X \in M_n(S)$

$\|A \cdot X \cdot B\|_n \leq \|A\|_{M_n(\mathbb{C})} \|X\|_n \|B\|_{M_n(\mathbb{C})}$

- $\|X \oplus 0\|_{m+n} = \|X\|_n$

...ebra.

^{n} with the "right" norm

$a_i^* a_i \| \left(\leq \sum_{i=1}^{n} \|a_i^* a_i\| \right)$

$V \hookrightarrow C(V^*)$

$M_n(V)$ —

A closed subspace!

For $S \subset A$ closed linear subspace $M_n(S)$ carries a canonical norm $\|\cdot\|_n$

Intrinsic definition of operator space:

Let S be a vector space with for each n a norm on $\|\cdot\|_n$ on $M_n(S)$, such that:

- $M_n(S)$ is Banach for $\|\cdot\|_n$.
- for all $A, B \in M_n(\mathbb{C})$ and $X \in M_n(S)$

$$\|A \cdot X \cdot B\|_n \leq \|A\|_{M_n(\mathbb{C})} \|X\|_n \|B\|_{M_n(\mathbb{C})}$$

- $\|X \oplus 0\|_{m+n} = \|X\|_n$

$$X \oplus 0 = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+m}(S)$$

$X \in M_n(S)$.

Thm. If S is an abstract operator space the there exists a C^*-algebra A and an inclusion $J: S \hookrightarrow A$ such that for all n

$$M_n(S) \to M_n(A)$$
$$(s_{ij}) \mapsto (J(s_{ij}))$$

are isometric.

The isomorphism $C^*(a) \simeq C_0(\sigma(a$...

maps a to $f(t) = t$. ...

Fact $\sigma(f(a)) = \{ f(\lambda)$...

Using the functions

$$f^+(x) = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

$$f^-(x) = \begin{cases} 0 & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

We have for $a = a^*$ $\quad a = f^+(a) - f$...

$= h - k$

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Completely bounded maps

Def Let S_1, S_2 be operator spaces. A linear map $\phi: S_1 \to S_2$ is completely bounded, if we have $\sup_n \|\phi_n\|_n < \infty$ where $\phi_n: M_n(S_1) \to M_n(S_2)$ is the map $\phi_n((s_{ij})) := (\phi(s_{ij}))$. A map ϕ is completely contractive if $\sup \|\phi_n\|_n \leq 1$. Lastly $\phi: S_1 \to S_2$ is completely positive if $\phi_n: M_n(S_1) \to M_n(S_2)$ is positive (For $S_1 \subset A_1$, $S_2 \subset A_2$ A_i C^*-algebras)

Example. The transpose $M_2(\mathbb{C}) \to M_2(\mathbb{C})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is contractive but not completely contractive. Also it is positive, but not completely positive.

$\phi_2: M_2(M_2(\mathbb{C})) \to M_2(M_2(\mathbb{C}))$ has norm 2

For (non)-positivity consider

$$\underset{\text{positive}}{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}} \xrightarrow{\phi_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow \text{not positive.}$$

On

On $B(\ell^2(\mathbb{N})) \cong \{(\lambda_{ij}) \mid i,j \in \mathbb{N}\}$.

$$(\lambda_{ij}) \longmapsto (\lambda_{ji})$$

Transpose is not completely bounded.

Complete boundedness is "the right" in several contexts. For instance for tensor products of C^*-algebras

$A_1 \bar{\otimes} A_2$ C^*-tensor product.

$\phi_1 : A_1 \to B_1$ and $\phi_2 : A_2 \to B_2$, bounded

then $\phi_1 \otimes \phi_2 : A_1 \bar{\otimes} A_2 \longrightarrow B_1 \bar{\otimes} B_2$ can be unbounded.

$\phi_1 \otimes \phi_2$ is completely bounded whenever ϕ_1 and ϕ_2 are completely bounded.

Dilations: H Hilbert space and $V : H \longrightarrow H$ is a linear isometry (that is, $V^*V = 1$ and VV^* is a projection)

Then $U : \begin{pmatrix} V & (1-VV^*) \\ 0 & V^* \end{pmatrix} : H \oplus H \longrightarrow H \oplus H$

is unitary. $U^* = \begin{pmatrix} V^* & 0 \\ 1-VV^* & V \end{pmatrix}$ and $U^*U = UU^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$P : H \oplus H \longrightarrow H \oplus H$

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \longmapsto \begin{pmatrix} h_1 \\ 0 \end{pmatrix}$$

$$PUP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V & 1-VV^* \\ 0 & V^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix}$$

$$PU^nP = V^n$$

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Isometric dilation of a contraction:

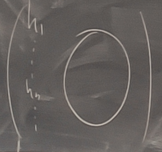
$T: H \to H$ is a contraction, $\|T\| \leq 1$

then $1 - T^*T \geq 0$ since $T^*T \leq \|T\| \leq 1$

So $(1-T^*T)^{1/2}$ is well-defined.

Now form $\ell^2(H) = \bigoplus_{n \in \mathbb{N}} H = \{(h_1, h_2, \ldots) \mid h_i \in H, \sum_{i=1}^{\infty} \|h_i\|^2 < \infty\} = \ell^2 \hat{\otimes} H$

$\langle (h_1, h_2, \ldots), (k_1, k_2, \ldots) \rangle = \sum_{i=1}^{\infty} \langle h_i, k_i \rangle$



$V: \ell^2(H) \to \ell^2(H)$

$h = (h_1, h_2, \ldots) \mapsto (Th_1, (1-T^*T)^{1/2} h_1, h_2, h_3, \ldots)$

$\langle V(h), V(h) \rangle = \sum_{i=1}^{\infty} \langle Th_1, Th_1 \rangle + \langle (1-T^*T)^{1/2} h_1, (1-T^*T)^{1/2} h_1 \rangle + \langle h_2, h_2 \rangle + \ldots$

$= \langle h_1, h_1 \rangle + \langle h_2, h_2 \rangle + \ldots$

$P_H: \ell^2(H) \to \ell^2(H)$

$(h_1, h_2, \ldots) \mapsto (h_1, \ldots)$

Then $P_H V^n P_H = T^n$

Thm. Let $T: H \to H$ be a contraction. Then there exists a Hilbert space K containing H and projection $P_H: K \to K$ such $P_H(K) = H$ and a unitary $U: K \to K$ such that $T^n = P_H U^n P_H$.

Corollary (von Neumann

Let $T: H \to H$ be

$h_3, \ldots)$

$)^{1/2} h_{4}, (1-T^*T)^{1/2} h_4 \rangle$

$_2 \rangle + \ldots$

$P_H : \ell^2(H) \longrightarrow \ell^2(H)$

$(h_1, h_2, \ldots) \mapsto (h_1, \ldots)$

Then $P_H V^n P_H = T^n$

Thm. Let $T: H \to H$ be a contraction. Then there exists a Hilbert space K containing H and projection $P_H : K \to K$ such $P_H(K) = H$ and a unitary $U: K \to K$ such that $T^n = P_H U^n P_H$.

Corollary (von Neumann's inequality)
Let $T: H \to H$ be a contraction and p a polynomial. Then

$$\|p(T)\| \le \sup\{|p(z)| \mid z \in S^1\}$$

Proof Let U be the unitary dilation then $p(T) = P_H\, p(U) P_H$ and

$$\|p(T)\| = \|P_H\, p(U) P_H\| \le \|p(U)\|$$

$$\|p(U)\| = \sup\{|p(z)| \mid z \in \sigma(U)\}$$
$$\le \sup\{|p(z)| \mid z \in S^1\}.$$

$P : H \oplus H \longrightarrow H \oplus H$

$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \longmapsto \begin{pmatrix} h_1 \\ 0 \end{pmatrix}$

$PUP = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V & 1-VV^* \\ 0 & V^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix}$

$PU^nP = V^n$