Chapter 2 : positive maps

Def: let A be a C^*-algebra. A subspace $S \subseteq A$ is called an *operator system* if it is self-adjoint, that is $S^* = S$, $1 \in S$

A linear map $\phi: A \to B$ between two linear spaces is positive if it maps positive elements to positive elements.

Proposition let A, B be unital C^*-algebras, $S \subset A$ an operator system. If $\phi: S \to B$ is positive, then it is bounded and

$$\|\phi\| \leq 2 \|\phi(1)\|_B$$

Proof If S is an operator system and $h \in S$ s.adjoint.

$$h = \frac{1}{2}\underbrace{(\|h\| \cdot 1 + h)}_{\in S_+} - \frac{1}{2}\underbrace{(\|h\| \cdot 1 - h)}_{\in S_+}$$

$$\Rightarrow \phi(h) = \frac{1}{2}\phi(\|h\| \cdot 1 + h) - \frac{1}{2}\phi(\|h\| \cdot 1 - h)$$

two positive elements

$$\Rightarrow \|p_1 - p_2\| \leq \max\{\|p_1\|, \|p_2\|\}$$

$$\Rightarrow \|\phi(h)\| \leq \frac{1}{2}\max\{\|\phi(\|h\| \cdot 1 + h)\|, \|\phi(\|h\| \cdot 1 - h)\|\}$$

$$\leq \|h\| \|\phi(1)\|$$

for a arbitrary, write $a = h + ik$, $\|h\|, \|k\| \leq \|a\|$
$h = h^*$, $k = k^*$

$$\Rightarrow \|\phi(a)\| \leq \|\phi(h)\| + \|\phi(k)\| \leq 2\|\phi(1)\| \|a\| \qquad \square$$

Example (Arveson) : let $\mathbb{T} \subseteq \mathbb{C}$ unit circle, $C(\mathbb{T})$ the C^*-algebra of continuous functions on $\mathbb{T}$, z the coordinate fct.

$S \subseteq C(\mathbb{T})$ subspace spanned by $1, z, \bar{z}$.

$$\phi(a + bz + c\bar{z}) := \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}$$

ϕ is a positive map. Indeed $a \cdot 1 + bz + c\bar{z}$ is positive
$\Leftrightarrow c = \bar{b}$ & $a \geq 2|b|$

$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in M_2(\mathbb{C})$ positive $\Leftrightarrow \alpha, \delta, \alpha\delta - \beta\gamma \underset{\geq 0}{\in \mathbb{R}}$

$\Rightarrow \phi$ is a positive map

$$2\|\phi(1)\| = 2 = \|\phi(z)\| \leq \|\phi\| \quad \Rightarrow \quad \|\phi\| = 2\|\phi(1)\|$$

Theorem (2.4)

let B a C^*-algebra, $\phi : C(X) \to B$ positive.

Then $\|\phi\| = \|\phi(1)\|$

Proof WLOG assume $\phi(1) \leq 1$. let $f \in C(X)$. $\|f\| \leq 1$.

let $\varepsilon > 0$ given. choose a finite open covering $\{U_i\}_{i=1}^{n}$ of X

s.t $|f(x) - f(x_i)| < \varepsilon$ for $x \in U_i$.

let p_i be a partition of unity subordinate to U [eg $\{p_i\}$ non neg
$\sum p_i = 1$
$p_i(x) = 0$
$x \notin U_i$]

put $\lambda_i = f(x_i)$.

Note: if $p_i(x) \neq 0 \Rightarrow x \in U_i \Rightarrow |f(x) - \lambda_i| < \varepsilon$

$\Rightarrow \forall x \in X$ we have

$$|f(x) - \sum \lambda_i p_i(x)| = |\sum (f(x) - \lambda_i) p_i(x)|$$

$$\leq \sum |f(x) - \lambda_i| \underbrace{p_i(x)} < \underbrace{\sum \varepsilon p_i(x)} = \varepsilon$$

no need for | |
because p_i non-negative

because part. of unity

$\to$ Lemma 2.3 let A be a C^*-alg, $p_i \in A$ positive $\sum p_i \leq 1$

if λ_i are scalars with $|\lambda_i| \leq 1$

$$\Rightarrow \left\| \sum_{i=1}^{n} \lambda_i p_i \right\| \leq 1$$

In our case $\sum |\lambda_i \phi(p_i)| \leq 1$, so that lemma 2.3 implies

$$\|\phi(f)\| \leq \|\phi(f - \sum \lambda_i p_i)\| + \|\sum \lambda_i \phi(p_i)\| < 1 + \varepsilon \|\phi\|$$

$$\underset{\varepsilon \text{ arbitrary}}{\Rightarrow} \quad \|\phi\| \leq 1$$ $\square$

we now need a classical results from complex analysis:

Lemma (Fejér - Riesz)

let $\tau(e^{i\theta}) = \sum_{n=-N}^{N} a_n e^{in\theta}$ strictly positive function on $\mathbb{T}$.

$\Rightarrow \exists$ polynomial $p(z) = \sum_{n=0}^{N} p_n z^n$ s.t

$$\tau(e^{i\theta}) = |p(e^{i\theta})|^2$$

(cf proof of Lemma 2.5)

Thm let T be an operator on a Hilbert space $\mathcal{H}$ with $\|T\| \le 1$

let $S \subseteq C(\mathbb{T})$ be an operator system defined by

$$S = \{ p(e^{i\theta}) + \overline{q(e^{i\theta})} \; : \; p, q \text{ polynomials} \}$$

The map $\phi : S \to B(\mathcal{H})$ defined by $\phi(p + \overline{q}) = p(T) + q(T)^*$

is positive.

Proof enough to prove positivity for strictly positive τ.

if $\tau \ge 0 \Rightarrow \tau + \epsilon 1$ is strictly positive $\forall \epsilon > 0$

$$\Rightarrow \phi(\tau) + \epsilon I = \phi(\tau + \epsilon \cdot 1) \ge 0 \quad \forall \epsilon$$

$$\phi(1) = 1$$

So we look at τ strictly positive

$$\Rightarrow \tau(e^{i\theta}) = \sum_{l,k=0}^{n} \alpha_l \overline{\alpha_k} e^{i(l-k)\theta}$$

$$\phi(\tau) = \sum_{l,k=0}^{n} \alpha_l \overline{\alpha_k} \, T(l-k)$$

with convention $T(j) = \begin{cases} T^j & j \ge 0 \\ (T^*)^{-j} & j < 0 \end{cases}$

Fix $x \in \mathcal{H}$

$$\langle \phi(\tau) x, x \rangle = \left\langle \begin{bmatrix} 1 & T^* & \cdots & T^{*n} \\ T & 1 & T^* & \\ \vdots & T & \ddots & T^* \\ T^n & \cdots & T & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \overline{\alpha_1} x \\ \vdots \\ \overline{\alpha_n} x \end{bmatrix}}_{\in \mathcal{H}^{(n)}} , \underbrace{\begin{bmatrix} \overline{\alpha_1} x \\ \vdots \\ \overline{\alpha_n} x \end{bmatrix}}_{\in \mathcal{H}^{(n)}} \right\rangle$$

If we can show this matrix operator is positive, we are done

let $R = \begin{bmatrix} 0 & & 0 \\ T & \ddots & \\ 0 & T & 0 \end{bmatrix}$ nihilpotent: $R^{n+1} = 0$ $\quad \|R\| \le 1$

$$1 + R + R^2 + \cdots + R^n + R^* + \cdots + (R^*)^n = \underbrace{(1-R)^{-1} + (1 - R^*)^{-1}}_{\text{because nihilpotent}} - I$$

So we need to check $(1-R)^{-1} + (1-R^*)^{-1} - I$ positive

let $h \in \mathcal{H}^{(n)}$, $h = (1-R)y$ for $y \in \mathcal{H}^{(n)}$

$$\langle ((1-R)^{-1} + (1-R^*)^{-1} - 1) h, h \rangle = \langle y, (1-R)y \rangle + \langle (1-R)y, y \rangle$$
$$- \langle (1-R)y, (1-R)y \rangle =$$
$$= \|y\|^2 - \|Ry\|^2 \ge 0$$

because R contraction $\square$

Corollary If A is a unital C^*-algebra, $a \in A$ with $\|a\| \leq 1$, then there is a unital positive map $\phi: C(\mathbb{T}) \to A$ with $\phi(p) = p(a)$.

Corollary let B, C C^*-algebras with unit, $A \subseteq B$ a subalgebra, $S = A + A^*$. If $\phi: S \to C$ positive, then $\|\phi(a)\| \leq \|\phi(1)\| \|a\|$ $\forall a \in A$

Corollary (Russo - Dye).
Let A, B be unital C^*-algebras, $\phi: A \to B$ a positive map.
Then $\|\phi\| = \|\phi(1)\|$.

We conclude by showing how positive maps arise:

Lemma let A be a C^*-algebra, $S \subseteq A$ an operator system, and $f: S \to \mathbb{C}$ a linear functional with $f(1) = 1$, $\|f\| = 1$.

If $a \in S$ is a normal element of a, then $f(a)$ lies in the closed convex hull of the spectrum of a.

Proof Recall that the C° of a compact set is the intersection of all closed disks containing the set.

Suppose $f(a) \notin C^\circ(\sigma(a))$. Then $\exists\ \lambda$ and $r>0$ s.t. $|f(a) - \lambda| > r$

while $\sigma(a) \subseteq \{z : |z - \lambda| \leq r\} \subseteq \mathbb{C}$

But then $\sigma(a - \lambda 1) \subseteq \{z : |z| \leq r\}$

moreover for a normal $\|a\| = spr(a) \Rightarrow \|a - \lambda 1\| \leq r$

while $|f(a) - \lambda| > r$ $\lightning$ with $\|f\| = 1$

$\square$

Remark: the convex hull of the spectrum of a positive operator $\subseteq \mathbb{R}_+$, so f must be positive.

Proposition: Let S be an operator system, B a unital C^*-algebra $\phi: S \to B$ a unital contraction. Then ϕ is positive.

Pf: By GNS, we represent B on a Hilbert space $\pi: B \to B(H)$

let $x \in H$, $\|x\| = 1$, set $f(a) = \langle \pi\, \phi(a) x, x \rangle$

$\Rightarrow f(1) = 1$, $\|f\| \leq \|\phi\|$. By the previous lemma, if a positive, $f(a)$ positive $\underset{\uparrow x \text{ arbitrary}}{\Rightarrow}$ $\phi(a)$ positive

$\square$

Proposition let A be a unital C^*-algebra, $\mathcal{M} \subseteq A$ subspace with $1 \in \mathcal{M}$.
If B is a unital C^*-algebra $\phi : \mathcal{M} \to B$ a unital contraction, then

$$\tilde{\Phi} : \mathcal{M} + \mathcal{M}^* \longrightarrow B \qquad \text{given by}$$

$$\phi(a + b^*) = \phi(a) + \phi(b)^* \quad \forall$$

is well-defined & there is a unique positive extension of Φ to $M + M^*$

Proof: Suppose Φ has a positive extension, then by s.adjointness of positive maps the extension must satisfy $\forall$

To prove well-definedness: $a, a^* \in \mathcal{M} \Rightarrow \phi(a^*) = \phi(a)$

let $S_1 = \{a : a \in M \ \& \ a^* \in \mathcal{M}\}$

$\Rightarrow S_1$ is an operator system by definition, and since Φ is unital + contractive on $S_1 \Rightarrow \Phi$ is positive $\Rightarrow$ ϕ is self-adjoint on S^1
(Exercise 2.1.)

$\Rightarrow \tilde{\Phi}$ is well-defined.

To see that $\tilde{\Phi}$ is positive we represent $B \subseteq B(H)$, define $\tilde{\rho}(a) := \langle \tilde{\Phi}(a)x, x \rangle$
to prove $\tilde{\rho}$ positive we use Hahn-Banach.

Indeed, let $\rho : M \to \mathbb{C}$ $\rho(a) = \langle \phi(a)x, x \rangle$, $\|\rho\| = 1$

$\Rightarrow \rho$ extends to $\rho_1 : M + \mathcal{M}^* \to \mathbb{C}$ $\|\rho_1\| = 1$
HB

$\Rightarrow \rho_1$ positive, so $\rho_1(a + b^*) = \rho(a) + \overline{\rho_1(b)} = \tilde{\rho}(a + b^*)$

$\Rightarrow \tilde{\rho}$ positive