Completely Positive Maps

Let A be C^*-algebra and let $M \subseteq A$ be subspace, then we call M an operator space

$\phi : S \longrightarrow B$ is positive if $x \geq 0 \Rightarrow \phi(x) \geq 0$

We say that ϕ_n is n-positive if ϕ_n is positive. We call ϕ completely positive if it is n-positive for all n

ϕ is completely isometric (contractive) if ϕ_n is isometric (contractive) $\forall n$.

ϕ is completely bounded if $\sup_n \|\phi_n\| < \infty$.

$k \leq n \Rightarrow \|\phi_k\| \leq \|\phi_n\|$

Lemma 3.1

Let A be a unital C^*-algebra $a, b \in A$, then

(i) $\|a\| \leq 1 \iff \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \geq 0$

(ii) $a^*a \leq b \iff \begin{bmatrix} 1 & a \\ a^* & b \end{bmatrix} \geq 0$

Proof of (ii)

Take $\pi : A \longrightarrow B(H)$, $A = \pi(a)$, $B = \pi(b)$

$A^*A \leq B$.

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 1 & A \\ A^* & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = \langle x, x \rangle + \langle x, Ay \rangle + \langle Ay, x \rangle + \langle y, By \rangle$$

$$= \langle x + Ay, x + Ay \rangle + \langle y, (B - A^*A) y \rangle$$

$B \not\geq A^*A$ then $\exists y \in H$ s.t. $\langle y, (B - A^*A) y \rangle = \lambda + \mu i$

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 1 & A \\ A^* & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = \langle x + Ay, x + Ay \rangle + \lambda + \mu i \text{ , set } x = -Ay$$

$\square$

Propositio 3.2

S operator system, B C^*-algebra with unit, $\phi : S \to B$ 2-positive and unital.

Then ϕ contractive.

Proof: $a \in S$, $\|a\| \leq 1 \Rightarrow \phi_2\left(\begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & \phi(a) \\ \phi(a)^* & 1 \end{bmatrix}$

Proposition 3.3

A, B unital C^*-algebras, $\phi : A \to B$ unital and 2-pos, then $\phi(a)^* \phi(a) \leq \phi(a^*a)$

Proof: $\begin{bmatrix} 1 & a \\ a^* & a^*a \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$

Completely Positive Maps

Proposition 3.4 A, B unital C^*-algebras

$M \subseteq A$ subspace, $1 \in M$, $S = M + M^*$.

If $\phi : M \to B$ is unital 2-contractive then the map $\tilde{\phi} : S \to B$, $\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$ is 2-positive and contractive

Proof: ϕ contractive $\Rightarrow$ $\tilde{\phi}$ well-defined (prop 2.12). $M_2(S) = M_2(M) + M_2(M)^*$ and $(\tilde{\phi})_2 = \widetilde{(\phi_2)}$.

ϕ_2 contractive $\Rightarrow$ $\tilde{\phi}_2$ positive (2.12)

$\Rightarrow$ $\tilde{\phi}$ contractive $\square$

Proposition 3.5 Same assumptions

ϕ is completely contractive, then $\tilde{\phi}$ is completely positive and completely contractive

$$M_2(M_n(S)) = M_{2n}(S)$$

norms inherited from $M_2(M_n(A)) \cong M_{2n}(A)$

Completely Positive Maps

Examples: – $\phi : A \to B$ $*$-homomorphism

$\phi_n : M_n(A) \to M_n(B)$ —— " ——

– Take A C^*-algebra and fix $x, y \in A$

$\phi : A \to A$, $\phi(a) = xay$.

$\phi_n((a_{ij})) = (x a_{ij} y) = \mathrm{diag}(x) \cdot (a_{ij}) \cdot \mathrm{diag}(y)$

$\|\phi_n\| \le \|x\| \cdot \|y\| \Rightarrow \phi$ is cb

if $x = y^*$, then ϕ is CP

– H_1, H_2 Hilbert spaces, $V_i \in B(H_1, H_2)$ $i = 1, 2$

$\pi : A \to B(H_2)$, then $\phi : A \to B(H_1)$

$a \mapsto V_2^* \pi(a) V_1$

Proposition 3.6 $S \subseteq A$ operator system, B C^*-algebras and $\phi : S \to B$ CP

$\Rightarrow \phi$ cb and $\|\phi(1)\| = \|\phi\| = \|\phi\|_{cb}$

$\|\phi(1)\| \le \|\phi\| \le \|\phi\|_{cb}$ clear

WTS: $\|\phi\|_{cb} \le \|\phi(1)\|$

$A=(a_{ij})\in M_n(S)$, $\|A\|\le 1$

$$\begin{bmatrix}1 & A\\ A^* & 1\end{bmatrix}\ge 0 \Rightarrow \begin{bmatrix}\phi_n(I) & \phi_n(A)\\ \phi_n(A^*) & \phi_n(I)\end{bmatrix}\ge 0$$

$\square$

$A,B\in M_n$, $A=(a_{ij})$, $B=(b_{ij})$

$A*B=(a_{ij}\cdot b_{ij})$. $S_A: M_n\to M_n$, $B\mapsto A*B$

$V:\mathbb{C}^n\to\mathbb{C}^n\otimes\mathbb{C}^n$, $e_i\mapsto e_i\otimes e_i$

$S_A(B)=V^*(A\otimes B)V$

<u>Proposition 3.8</u> S operator space and $f:S\to\mathbb{C}$ bounded linear functional. Then $\|f\|_{cb}=\|f\|$. Furthermore, if S is an operator system, then $f\ge 0 \Rightarrow f$ is CP

<u>Proof</u>: $(a_{ij})\in M_n(S)$, $x,y\in\mathbb{C}^n$, $\|x\|=\|y\|=1$

$$|\langle f((a_{ij}))x,y\rangle| = \left|\sum f(a_{ij})x_j\bar{y}_i\right| = \left|f\left(\sum a_{ij}x_j\bar{y}_i\right)\right|$$

$$\le \|f\|\cdot\|\underbrace{\textstyle\sum a_{ij}x_j\bar{y}_i}_{=z}\|$$

Positive Maps

Completely Positive Maps

z is the $(1,1)$ entry of

$$\begin{bmatrix} \bar{y}_1 \cdot 1 & \bar{y}_2 \cdot 1 & \cdots & \bar{y}_n \cdot 1 \\ & 0 & & \end{bmatrix} (a_{ij}) \begin{bmatrix} x_1 \cdot 1 & \\ x_2 \cdot 1 & \\ \vdots & 0 \\ x_n \cdot 1 & \end{bmatrix} \quad (*)$$

when $y = x$ and $(a_{ij}) \geq 0$ $\square$

Rmk: if $\phi : S \to C(X)$, the same result holds $\phi^x : S \to \mathbb{C}$, $\phi^x(a) = \phi(a)(x)$

Lemma 3.10 B C^*-algebra and $(p_{ij}) \in M_n$ and $q \in B^+$ are positive. Then $(q \cdot p_{ij}) \in M_n(B)$ is positive

proof: $\mathrm{diag}(q^{1/2})(p_{ij})\mathrm{diag}(q^{1/2})$, $(p_{ij}) = x^* x$

$= (\mathrm{diag}(q^{1/2}) x^*) x \,\mathrm{diag}(q^{1/2}) \geq 0$

Theorem 3.11 B C^*-algebra and $\phi : C(X) \longrightarrow B$ is positive. Then it is CP.

This allows us to generalize Von Neumann's inequality to matrices of polynomials: If $\|T\| \leq 1$

$$\|(p_{ij}(T))\|_{B(H^n)} \leq \|(p_{ij})\|_{M_n(C(\mathbb{T}))}$$

The numerical radius of $T \in B(H)$:
$$\sup_{\substack{x \in H \\ \|x\|=1}} |\langle Tx, x\rangle| =: w(T)$$

Theorem 3.15 $T \in B(H)$ and $S \subseteq C(\mathbb{T})$

$S = \{p + \bar{q} : p, q \text{ polynomials}\}$ TFAE

(i) $w(T) \leq 1$

(ii) $\phi : S \to B$ defined by $\phi(p + \bar{q}) = p(T) + q(T)^* + (p(0) + \overline{q(0)})I$ is positive

Completely Positive Maps

$w(T) \le 1 \Rightarrow \|\phi\|_{cb} = \|\phi(1)\| = \|2+1\| = 2$

$\|p(T)\| \le \|\phi(p) - p(0)1\| \le 3\|p\| \Rightarrow p \underset{\text{continuous}}{\mapsto} p(T)$

Corollary 3.16 $T \in B(H)$ and $w(T) \le 1$, $f \in A(D)$ with $f(0)$. Then $w(f(T)) \le 1$

defn: $A(D) = \{f : D \to \mathbb{C}$, holomorphic & extend continuously to $\overline{D}\}$

Lemma 3.20 B C^*-algebra and $(p_{ij}) \in M_n$ and $q \in B^+$ are positive. Then $(q \cdot p_{ij}) \in M_n(B)$ is positive

proof: $\mathrm{diag}(q^{1/2})(p_{ij})\mathrm{diag}(q^{1/2})$, $(p_{ij}) = x^*x$

$= (\mathrm{diag}(q^{1/2}$