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Chapter 3, completely positive maps

let A be a C^*-algebra, $\mathcal{M} \subseteq A$ a subspace. then we call $\mathcal{M}$ an operator space. $M_n(\mathcal{M}) \subseteq M_n(A)$ inherits the norm structure of $M_n(A)$.

$S \subseteq A$ operator system. $\phi: S \to B$ is positive if $x \geq 0 \Rightarrow \phi(x) \geq 0$.

- we say that ϕ is n-positive if $\phi_n: M_n(S) \to M_n(B)$ is positive;
- we say that ϕ is completely positive if ϕ_n is positive for all $n \in \mathbb{N}$.
- ϕ is completely isometric / contractive if ϕ_n is isometric / contractive $\forall n$.
- ϕ is completely bounded if $\sup_n \|\phi_n\|$ is finite (bounded $\forall n$ not enough)

$$\Rightarrow \|\phi\|_{cb} = \sup_n \|\phi_n\|$$

Remark ϕ is n bounded $\Rightarrow$ ϕ is $(n-1)$ bounded.

Lemma (3.1) let A be a C^*-algebra with unit. let $a, b \in A$. then

(i) $\|a\| \leq 1 \iff \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}$ positive in $M_2(A)$

(ii) $\begin{bmatrix} 1 & a \\ a^* & b \end{bmatrix}$ positive in $M_2(A) \iff a^*a \leq b$

(ii) $\Rightarrow$ (i), so we prove (ii)

'$\Leftarrow$' Take $\pi: A \to B(\mathcal{H})$ a faithful rep, set $\alpha = \pi(a)$, $\beta = \pi(b) \Rightarrow \alpha \leq \beta$ positivity for HS operators.

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 1 & \alpha \\ \alpha^* & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \langle x, x \rangle + \langle y, \alpha^* x \rangle + \langle x, \alpha^* y \rangle + \langle y, \beta y \rangle$$

$$= \langle x + \alpha y, x + \alpha y \rangle + \langle y, (\beta - \alpha^* \alpha) y \rangle$$

$$\geq 0 \qquad \text{if} \quad \beta - \alpha^* \alpha \geq 0$$

'$\Rightarrow$' assume $\beta < \alpha^* \alpha$

$\Rightarrow \exists y$ s.t $\langle y, (b - \alpha^* \alpha) y \rangle = \lambda + \mu i$

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 1 & \alpha \\ \alpha^* & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \langle x + \alpha y, x + \alpha y \rangle + \lambda + \mu i \qquad \text{set } x = -ay \quad \lightning$$

$\square$

<u>Proposition 3.2</u> let S be an operator system, B a C^*-algebra with unit, $\phi : S \to B$ 2 positive and unital.

Then Φ is contractive.

<u>Proof</u>: $a \in S$, $\|a\| \le 1 \Rightarrow \phi_2\left(\begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & \phi(a) \\ \phi(a)^* & 1 \end{bmatrix}$ positive

$\Rightarrow \|\phi(a)\| \le 1$
lemma 3.1

<u>Proposition 3.3</u> (Schwartz inequality for 2 positive maps / Kadison's inequality)

let A, B be unital C^*-algebras, $\phi : A \to B$ a unital 2-positive map. Then $\phi(a)^* \phi(a) \le \phi(a^* a)$ $\forall a \in A$

(follows from 3.1 (ii))

<u>Proposition 3.4</u> let A, B unital C^*-algebras, $M \subseteq A$ subspace $1_A \in M$

let $S = M + M^*$. If $\phi : M \to B$ unital & 2 contractive, then $\tilde{\phi} : S \to B$ is positive & 2-contractive.

$\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$

<u>Pf</u> by Prop 2.12, if ϕ is contractive $\Rightarrow \tilde{\phi}$ is well defined

moreover, $M_2(S) = M_2(M) + M_2(M)^*$ & $(\tilde{\phi})_2 = (\tilde{\phi_2})$

by 2.12 (again), we have ϕ_2 contractive $\Rightarrow \tilde{\phi}_2$ positive

$\Rightarrow \tilde{\phi}$ contractive
prop 3.2 $\square$

<u>Proposition 3.5</u> under the same conditions of 3.4, suppose $\phi : M \to B$ unital & completely contractive $\Rightarrow \tilde{\phi} : S \to B$ is completely positive & completely contractive.

<u>Remark</u> we have identified $M_{2n}(S)$ with $M_2(M_n(S))$

Algebraically: no-problem.

Norms are the same: $M_2(M_n(S))$ inherits from $M_2(M_n(A))$ } here we have an iso of C^*-algebras

$M_{2n}(S)$ " " $M_{2n}(A)$

Examples • $\phi : A \to B$ $*$-homomorphism between C^* algebras

$\Rightarrow \phi_n : M_n(A) \to M_n(B)$ also $*$ hom.

$*$ hom $\Rightarrow$ positive & contractive $\Rightarrow \Phi$ is completely positive & completely contractive

• Fix a C^*-algebra A, $x, y \in A$ & define $\phi : A \to A$ by $\phi(a) = xay$

let $(a_{ij}) \in M_n(A)$

$\Rightarrow \| \phi_n(a_{ij}) \| = \| (x a_{ij} y) \| = \| \mathrm{diag}(x) \cdot (a_{ij}) \, \mathrm{diag}(y) \| \leq \|x\| \, \|a_{ij}\| \, \|y\|$

$\Rightarrow \phi$ cb & $\|\phi\|_{cb} \leq \|x\| \cdot \|y\|$. if $x = y^*$ $\Rightarrow$ Completely positive

• combining the 2 examples:

let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces $V_1, V_2 : \mathcal{H}_i \longrightarrow \mathcal{H}_2$ two bounded operators.

let $\pi : A \to B(\mathcal{H}_2)$ be a $*$-homomorphism & define

$\phi : A \to B(\mathcal{H}_1)$ as
$a \mapsto V_2^* \pi(a) V_1$

Then Φ is completely bounded & $\|\phi\|_{cb} \leq \|V_1\| \|V_2\|$

moreover if $V_1 = V_2 \Rightarrow \phi$ is completely positive

Remark All completely positive maps we encountered are completely bounded. This is not a coincidence:

Proposition $S \subseteq A$ operator system. B a C^*-algebra.
let $\phi : S \to B$ completely positive.
Then ϕ is completely bounded and $\|\phi(1)\| = \|\phi\| = \|\phi\|_{cb}$.

Proof: clearly, $\|\phi(1)\| \leq \|\phi\| \leq \|\phi\|_{cb}$

let us prove $\|\phi\|_{cb} \leq \|\phi(1)\|$

let us consider a matrix $A = (a_{ij}) \in M_n(S)$ with $\|(a_{ij})\| \leq 1$

let us consider $\begin{bmatrix} 1_n & A \\ A^* & 1_n \end{bmatrix} \in M_{2n}(S)$ is positive (3.1)

$$\Rightarrow \quad \phi_{2n}\left(\begin{pmatrix} 1_n & A \\ A^* & 1_n \end{pmatrix}\right) = \left[\begin{bmatrix} \phi_n(I_n) & \phi_n(A) \\ \phi_n(A^*) & \phi_n(I_n) \end{bmatrix}\right]$$

positive $\Rightarrow \quad \|\phi_n(A)\| \le \|\phi_n(1_n)\| = \|\phi(1)\| \quad \forall n$

$\square$

Schur products & tensor products

let $A, B \in M_n$

$\Rightarrow \quad A = (a_{ij}) \quad , \quad B = (b_{ij}) \qquad \Rightarrow \qquad A * B = (a_{ij} \cdot b_{ij})_{ij}$

$\forall A$ we get a linear map $\quad S_A: M_n \to M_n \qquad B \mapsto A * B$

let $V: \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$
$\quad e_i \mapsto e_i \otimes e_i$

$$S_A(B) = V(\underbrace{A \otimes B}_{\text{Kronecker product}})V^*$$

Proposition 3.8 let S be an operator space and $f: S \to \mathbb{C}$ a bounded linear functional. then $\|f\|_{cb} = \|f\|$

Furthermore, if S is an operator system & f is positive $\Rightarrow$ f is completely positive

Proof let $(a_{ij}) \in M_n(S)$, $x, y \in \mathbb{C}^n$ with $\|x\| = \|y\| = 1$.

$$\Rightarrow \quad |\langle f(a_{ij})x, y\rangle| = \left|\sum_{ij} f(a_{ij}) x_j \overline{y_i}\right| = \left|f\left(\sum_{ij} a_{ij} x_j \overline{y_i}\right)\right|$$
$$\le \|f\| \left\|\sum a_{ij} x_j \overline{y_i}\right\|$$

our goal is to show $\quad \left\|\sum a_{ij} x_j \overline{y_i}\right\| \le \|(a_{ij})\|$

follows from it being $\quad$ (1,1) entry of the matrix product

$$\begin{bmatrix} \overline{y_1}\cdot 1 & \overline{y_2}1 \dots & \overline{y_n}1 \\ & 0 & \end{bmatrix}\begin{bmatrix} a_{11} & \dots & a_{1n} \\ & & \\ a_{n1} & & a_{nn} \end{bmatrix}\left[\begin{array}{c|c} x_1\cdot 1 & 0 \ \dots \\ \vdots & \ddots \\ x_n\cdot 1 & 0 \end{array}\right]$$

since x, y were chosen to have norm 1. (as vectors).

- to conclude f completely positive, let $y = x$

$$\langle f_n(a_{ij})x, x\rangle = f\left(\sum a_{ij} x_j \overline{x_i}\right) \qquad \text{positive} \Leftrightarrow a_{ij} \text{ positive}$$

(1,1) entry of a positive matrix (diag entries are positive)

Remark: if $\phi: S \to C(X)$ (comm + unital C^*-algebra) is a bounded linear map, then $\|\phi\|_{cb} = \|\phi\|$.
Furthermore, if S is an operator system & ϕ positive, then ϕ is completely positive. (cf. Theorem 3.9).

Lemma 3.10 (p_{ij}) positive scalar matrix. q be a positive element of some C^*-algebra. then $(q \cdot p_{ij})$ positive in $M_n(B)$.

Theorem (Stinespring)
let B a C^*-algebra, and let $\phi: C(X) \to B$ positive. Then Φ is completely positive

Proof: uses a partition of unity argument.

Corollary (matrix valued version of von Neumann inequality)
let T be an operator on a Hilbert space with $\|T\| \leq 1$. (p_{ij}) an $n \times n$ matrix of polynomials

$$\|(p_{ij}(T))\|_{B(H^n)} \leq \sup\{\|p_{ij}(z)\|_{M_n} : |z| = 1\}$$

Def: The numerical radius of $T \in B(H)$

$$\sup_{\substack{x \in H \\ \|x\| \leq 1}} |\langle Tx, x\rangle| =: \omega(T)$$

Thm $T \in B(H)$, $S \subseteq C(\mathbb{T}) = \{p + \bar{q} \mid p, q \text{ polynomials}\}$
Tfae
1. $\omega(T) \leq 1$
2. $\phi: S \to B(H)$ defined by
$$\phi(p + \bar{q}) = p(T) + q(T)^* + \overline{p(0) + q(0)} \cdot 1$$ is positive

This has the consequence that we can extend the functional calculus from polynomials to $A(\mathbb{D})$ for T with $\omega(T) \leq 1$

$\downarrow$
functions that are analytic on $\mathbb{D}$ & extend continuously to $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$

Corollary (Berger-Kato-Stampfli) $T \in B(H)$, $\omega(T) \leq 1$, $f \in A(\mathbb{D})$
$f(0) = 0 \to \omega(f(T)) \leq \|f\|$