Ch 4 Dilation theorems

1 Sz.-Nagy's dilation theorem.

Thm 1 (Stinespring's dilation theorem)

Let A be a unital C^*-algebra, let completely positive map $\phi : A \to B(H)$ there exists a (larger) Hilbert space K, and a unital $*$-homom. $\pi : A \to B(K)$ and $V : H \to K$ with $\|\phi(1)\|^2 = \|V\|^2$ s.t. $\phi(a) = V^* \pi(a) V$

proof: Consider $A \otimes H$. def symmetric bilinear form

$$\langle a \otimes x, b \otimes y \rangle = \langle \phi(b^* a) x, y \rangle_H$$

extend $\langle \cdot, \cdot \rangle$ linearly on $A \otimes H$.

$$\left\langle \sum_{j=1}^{n} a_j \otimes x_j , \sum_{i=1}^{n} a_i \otimes x_i \right\rangle = \left\langle \left(\phi(a_i^* a_j)\right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle$$

is positive semi-definite.

$$N = \{ u \in A \otimes H \mid \langle u, u \rangle = 0 \} = \{ u \in A \otimes H \mid \langle u, v \rangle = 0 \ \forall v \in A \otimes H \}$$

$$K = \overline{A \otimes H / N}^{\langle \cdot, \cdot \rangle}$$

$$\pi(a)\left(\sum_{i=1}^{n} a_i \otimes x_i\right) = \sum_{i=1}^{n} a a_i \otimes x_i$$

$$\begin{pmatrix} a_1^* a^* a a_1 & a_1^* a^* a a_2 & \cdots \\ \vdots & & \\ a_n^* a^* a a_1 & \cdots & a_n^* a^* a a_n \end{pmatrix} = \begin{pmatrix} a_1^* & \\ \vdots & 0 \\ a_n^* & \end{pmatrix} \cdot a^* a \, I \begin{pmatrix} a_1 & \cdots & a_n \\ & 0 & \end{pmatrix} \Rightarrow A^* a^* a A \leq \|a^* a\| \cdot (a_i^* a_j)$$

$$\Rightarrow \left\langle \pi(a)\left(\sum_{j=1}^{n} a_j \otimes x_j\right), \pi(a)\left(\sum_{i=1}^{n} a_i \otimes x_i\right)\right\rangle$$

$$= \|a\|^2 \cdot \left\langle \sum a_j \otimes x_j , \sum a_i \otimes x_i \right\rangle$$

$\Rightarrow \pi(a) N \subset N \Rightarrow \pi(a)$ is well-defined on K.

$$V : H \longrightarrow K$$

$$x \longmapsto 1 \otimes x + N$$

$$\|Vx\|^2 = \langle 1 \otimes x, 1 \otimes x \rangle = \langle \phi(1) x, x \rangle \leq \|\phi(1)\| \cdot \|x\|^2,$$

Finally, we observe

$$\langle \underline{V^* \pi(a) V} x, y \rangle = \langle \pi(a) 1 \otimes x, 1 \otimes y \rangle_H$$

$$= \langle \underline{\phi(a)} x, y \rangle \qquad \blacksquare$$

$$VH = H \quad \phi(a) = V^* \pi(a) V$$
$$\cap$$
$$K$$

$$= P_H \pi(a) V = P_H \pi(a)|_K$$

Def We call (π, V, K) Stinespring representation for ϕ

Def: We call (π, V, K) <u>minimal</u> if $K = \overline{\pi(A) VH}$

Prop. Let A be a unital C^*-algebra, let $\phi : A \longrightarrow B(H)$ a completely positive map, and let (π_i, V_i, K_i) $i = 1, 2$ be two S.R. Then $\exists$ unitary $U : K_1 \longrightarrow K_2$

Such that, $UV_1 = V_2$, $U \pi_1 U^* = \pi_2$ $(*)$

proof

$$U\left(\sum_{i=1}^{n} \pi_1(a_i) V_1 h_j \right) = \sum_{j=1}^{n} \pi_2(a_j) V_2 h_j$$

$$\| \sum_j \pi_1(a_j) V_1 h_j \|^2 = \sum_{i,j} \langle V_1^* \pi_1(a_i^* a_j) V_1 h_j, h_i \rangle$$
$$= \sum_{i,j} \langle \phi(a_i^* a_j) h_j, h_i \rangle$$
$$= \| \sum_j \pi_2(a_j) V_2 h_j \|^2 \quad \blacksquare$$

Theorem 4.3 (Sz - Nagy's dilation thm)

Let $T \in B(H)$, $\|T\| \leq 1$. Then there exists a Hilbert space $K \supset K' \supset H$ and a unitary U on K, with K is the smallest closed reducing space for U, containing H s.t.

$$T^n = P_H U^n|_H \quad \forall n \in \mathbb{N}$$

Proof: $\phi(p + \bar{q}) := p(T) + q(T)^*$ is completely positive from $C(\mathbb{T}) \to B(H)$. Thm 2.6, exe 2.2 + thm 3.11

$\exists (\pi, V, K)$ as a minimal S.R.

Since $\phi(1) = 1$, we may identity VH and H and setting $\pi(z) = U$, and

$$T^n = \phi(z^n) = P_H \pi(z^n) = P_H \pi(z)^n = P_H U^n$$

$K := \overline{\{ U^n H : n \in \mathbb{N} \}}$

X be a compact Hausdorff

Let X be a compact Hausdorff Space, $\mathcal{B}_{\mathbb{C}}$ as the σ-algebra of Borel sets

A $B(H)$-valued measure on X is a map $E: \mathcal{B} \longrightarrow B(H)$, that is weakly countably additive

$$\langle E(\dot{\bigcup_i} B_i)x, y\rangle = \sum_i \langle E(B_i)x, y\rangle$$

$\|E\| = \sup\{\|E(B)\|,\ B \in \mathcal{B}\}$. if $\|E\| < \infty$, call E bounded

Call E regular if $\mu_{x,y}(B) = \langle E(B)x, y\rangle$ is regular for all $x, y \in H$

<u>One to One corr</u>: Give a regular bounded $B(H)$-valued measure E,

$\phi_E : C(X) \longrightarrow B(H)$

$$\langle \phi_E(f)x, y\rangle = \int f \, d\mu_{x,y}$$

<u>Conversely</u>: $\phi: C(X) \longrightarrow B(H)$, $\phi(f) \in B(H)$

$\langle \phi(f)x, y\rangle$ is a <u>linear function</u>

by using Riesz-Markov-Kakutani repr thm

$\exists$ regular $\mu_{x,y}$ s.t. $\langle \phi(f)x, y\rangle = \int f \, d\mu_{x,y}$

by Riesz-repr $\Rightarrow \mu_{x,y}(B) = \langle E(B)x, y\rangle$

$B(H)$-valued measure is called

(i) Spectral if $E(M \cap N) = E(M) \cdot E(N)$

(ii) Positive if $E(M) \geq 0$

(iii) Self-adjoint if $E(M)^* = E(M)$

<u>Prop 4.5</u>. Let E be a regular bounded $B(H)$-valued measure and let $\phi : C(X) \longrightarrow B(H)$ be the corr linear map

(i) ϕ is homo iff E spectral

(ii) ϕ is posi iff E is positive

(iii) ϕ self-ad iff E self-ad

(iv) ϕ $*$-homo iff E spectral + self-ad.

$\forall x, y \in \mathcal{H}$, $\mu_{x,y}(M \cap N) = \int 1_{M \cap N} \, d\mu_{x,y}$

$$\begin{aligned} \langle E(M \cap N) x, y \rangle &= \int 1_M \cdot 1_N \, d\mu_{x,y} \\ &= \langle \phi(1_M) \phi(1_N) x, y \rangle \\ &= \langle E(M) E(N) x, y \rangle \end{aligned}$$

$\langle \phi(1_M) x, y \rangle$
$= \mu_{x,y}(M)$
$= \langle E(M) x, y \rangle$

$M \subset B$, $\exists \underset{\text{closed}}{A} \subseteq M$

$\mu_{x,y}(M \setminus A) \leq \varepsilon$

$\exists f(x) \in [0,1]$, $f|_A = 1$, $f|_{X \setminus M^c} = 0$

Theorem (Naimark)

Let E be a bounded regular positive $B(H)$-valued measure on X. Then there exists a Hilbert space K and a bounded linear operator $H \xrightarrow{V} K$, and a regular, self-ad. spectral $B(K)$-valued measure F on X such that

$$E(B) = V^* F(B) V$$

Proof: Let $\phi : C(X) \longrightarrow B(H)$ be the positive, linear map corresponding to E.

By thm 3.11 ϕ is completely positive.

Then, by Stinespring's dilation theorem, $\exists (\pi, V, K)$, $\pi : C(X) \longrightarrow B(H)$ s.t. $\phi(f) = V^* \pi(f) V$ for all $f \in C(X)$. ∎

$\phi \updownarrow E$, $\pi \updownarrow F$

Let G be a group. Let $\phi : G \longrightarrow B(H)$.

We call ϕ completely positive definite if, for $g_1, \dots, g_n \in G$, $\left(\phi(g_i^{-1} g_j)\right)$ is positive.

If G is a topological group, we call $\phi : G \longrightarrow B(H)$ weakly continuous, if $\{g_i\} \longrightarrow \{g\}$ then $\langle \phi(g_i)x, y\rangle \longrightarrow \langle \phi(g)x, y\rangle$

is unitary: let $f_1, f_2 \in C_0(G, H)$, then we have

$(\rho(g_2)^* f)(g) = f(g_2 g)$

$\Leftarrow$

$\langle \rho(g_2) f_1, f_2 \rangle = \sum_{g, g'} \langle \phi(g^{-1} g') f_1(g_2^{-1} g'), f_2(g) \rangle$

$g'' = g_2^{-1} g'$

$= \sum_{g, g''} \langle \phi(g^{-1} g_2 g'') f_1(g''), f_2(g) \rangle$

$g = g_2^{-1} g'''$

$= \sum_{g''', g''} \langle \phi(g'''^{-1} g'') f_1(g''), f_2(g_2 g''') \rangle_H$

$= \langle h, \sum_g \phi(g^{-1})^* f(g) \rangle_H$

$= \langle h, \sum_g \phi(g) f(g) \rangle_H$

$V^* \rho(g) V(h) = V^* \rho(g) (V(h))$

$= V^* (V h(g^{-1} \cdot)) =$

$= \phi(g)$

is called strongly continuous if $\| \phi(g_\lambda) x - \phi(g) x \| \to 0$

$*$- strongly ——— if $+ \| \phi(g_\lambda)^* x - \phi(g)^* x \| \to 0$

$\boxed{\phi(g) = \phi(g^{-1})^*}$ g, e, g^{-1} $\phi(g_i^{-1} g_j)$

Thm (Naimark) ... be a topo gp and let $\phi : G \to B(H)$ be ...ly continuous and completely positive definite. Then there exists a Hilbert space K and bounded operator $V : H \to K$ and a $*$-strongly continuous unitary repr $\rho : G \to B(H)$ s.t. $\phi(g) = V^* \rho(g) V$

Proof $C_0(G, H)$ = finitely supported functions from G to H

$f = \sum_{i=1}^{n} h_i \cdot g_i \quad (f(g_i) = h_i)$

Define the bilinear form $\langle f_1, f_2 \rangle = \sum_{g, g'} \langle \phi(g^{-1} g') f_1(g'), f_2(g) \rangle_H$

Since ϕ is completely positive definite, $\langle \cdot, \cdot \rangle$ is positive semi-definite. Then define

$$N = \{ \langle f, f \rangle = 0, f \in C_0(G, H) \}$$

Let $K = \overline{C_0(G, H) / N}^{\langle \cdot, \cdot \rangle}$. Define $V : H \to K$

$\rho : G \to B(K)$ by $(\rho(g) f)(g') = f(g^{-1} g')$ $h \mapsto h \cdot e$

- V is bounded $\|V(h)\|^2 = \langle Vh, Vh\rangle$

$= \langle \phi(e^{-1}e)h, h\rangle$

- V is linear

$= \langle \phi(e)h, h\rangle \le \|\phi(e)\| \, \|h\|^2$

- ρ is unitary: let $f_1, f_2 \in C_c(G, H)$, then we have

$(\rho(g_1)^* f)(g) = f(g_1 g)$

$\Leftarrow$

$\langle \rho(g_1) f_1, f_2\rangle = \sum_{g, g'} \langle \phi(g^{-1}g') f_1(g_1^{-1}g'), f_2(g)\rangle$

$g'' = g_1^{-1}g'$

$= \sum_{g, g''} \langle \phi(g^{-1}g_1 g'') f_1(g''), f_2(g)\rangle$

$g = g_1^{-1}g'''$

$= \sum_{g''', g''} \langle \phi(g'''^{-1} g'') f_1(g''), f_2(g_1 g''')\rangle_H$

$\underline{\phi(g) = V^* \rho(g) V}$ since

$\langle Vh, f\rangle = \sum_g \langle \phi(g^{-1}e)h, f(g)\rangle_H$

$= \langle h, \sum_g \underline{\phi(g^{-1})^*} f(g)\rangle_H$

$= \langle h, \sum_g \phi(g) f(g)\rangle_H$

$\to V^* \rho(g) \underline{V(h)} = V^* \rho(g)(V(h))$

$= V^*(Vh(g^{-1}\cdot)) =$

$= \phi(g)$

$= f(g_\lambda g)$

$\Leftarrow$

$\langle \rho(g_\lambda) f_1, f_2 \rangle = \sum_{g, g'} \langle \phi(g^{-1}g') f_1(g_\lambda^{-1} g'), f_2(g) \rangle$

$g'' = g_\lambda^{-1} g'$ $= \sum_{g, g''} \langle \phi(g^{-1} g_\lambda g'') f_1(g''), f_2(g) \rangle$

$g = g_\lambda^{-1} g'''$ $= \sum_{g''', g''} \langle \phi(g'''^{-1} g'') f_1(g''), f_2(g_\lambda g''') \rangle_{\mathcal{H}}$

$V^* \rho(g) V(h) = V^* \rho(g) (V(h))$

$= V^*(V_h(g^{-1} \cdot)) =$

$= \phi(g)$

Let $\{g_\lambda\}$ be the net that $\longrightarrow g_0$.
$\subset G$

Suffice to check $\rho(g_\lambda) \xrightarrow{\text{weakly}} \rho(g_0)$ on some dense subspace

$\sum \langle \rho(g_\lambda) f_1, f_2 \rangle = \sum_{g, g''} \langle \phi(g^{-1} g_\lambda g'') f_1(g''), f_2(g) \rangle_{\mathcal{H}}$

$= \langle \rho(g_0) f_1, f_2 \rangle$ ∎

Def (positive definite).

We call the map $G \longrightarrow B(H)$, p.d. if $\forall g_1, \dots, g_n \in G$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, $\sum_{i,j} \overline{\alpha_i} \alpha_j \phi(g_i^{-1} g_j)$ is posi.

$\{$unitary repr $\rho: \mathbb{Z}^n \longrightarrow B(H)\}$ $\quad J = (j_1, \dots, j_n)$, $\rho(J)$

$\Updownarrow$ $\qquad \|$

$\{$ $*$-homo $\pi: C(\mathbb{T}^n) \longrightarrow B(H)\}$ $\quad \pi(z^J)$, $z^J = z_1^{j_1} \cdots z_n^{j_n}$

Prop 4.9 Let $\phi: \mathbb{Z}^n \longrightarrow B(H)$ be (completely) posit defini

Then $\exists$ uniquely (completely) positive defin map $\psi: C(\mathbb{T}^n) \to B(H)$ given by $\psi(z^J) = \phi(J)$

if $\gamma: H \longrightarrow \mathbb{C}$, $|\gamma(h)| \le N \cdot \|h\|^2$ $\quad \gamma(x) = \langle Tx, x \rangle$ $\Rightarrow \|T\| < \infty$

$Q(x,y) = \frac{1}{4}\big(\gamma(x+y) - \gamma(x-y) + i\gamma(x+iy) - i\gamma(x-iy)\big)$

$\gamma(x+y) + \gamma(x-y) = 2(\gamma(x) + \gamma(y))$