Cor 5.2. $V_1, \dots, V_n$ commuting isometries,

Cor 5.3. $V_1, \dots, V_n$ commuting isometries

Commuting Contractions on Hilbert spaces

Thm 5.1 $V_1, \dots, V_n$ commuting isometries $\Rightarrow \exists\, \mathcal{U} \supseteq H$ Hilbert space, $U_1, \dots, U_n$ commuting unitaries on $\mathcal{U}$ s.t. $V_1^{m_1} \cdots V_n^{m_n} = P_H U_1^{m_1} \cdots U_n^{m_n}|_H$

Proof Let U_1 on $\mathcal{U}_1$ be the minimal unitary dilation of V_1 $\Rightarrow \mathcal{U}_1 = \overline{\operatorname{span}}(\{U_1^n H \; n \in \mathbb{Z}\})$

$W_i : \mathcal{U}_1 \to \mathcal{U}_1, \; \sum_{n=-N}^{N} U_1^n h_n \longmapsto \sum_{n=-N}^{N} U_1^n V_i h_n$

W_i well-defined isometry, $\{U_1, W_2, \dots, W_n\}$ commute and $V_1^{m_1} \cdots V_n^{m_n} = P_H U_1^{m_1} W_2^{m_2} \cdots W_n^{m_n}|_H$

repeat procedure $\rightsquigarrow \square$

$$D = \begin{pmatrix} \rho(p_1) & & \\ & \ddots & \\ & & \rho(p_n) \end{pmatrix} \overset{\text{Naimark}}{\rightsquigarrow} \exists K \supseteq H,\ \pi: G \to B(K) \text{ unitary rep,}$$

$V: H \to K$, $\phi(g) = V^*\pi(g)V$, $1 = \phi(e) = V^*V$, $VH \simeq H$ $\square$

Cor 5.2: $V_1, \dots, V_n$ commuting isometries, p_{ij}, $i,j = 1, \dots, m$ polynomial in n variables

Then $\|(p_{ij}(V_1, \dots, V_n))\|_{B(H^{(m)})} \leq \sup\{\|(p_{ij}(z_1, \dots, z_n))\|_{M_m} \mid |z_k| \leq 1, 1 \leq k \leq n\}$

Proof: $p(V_1, \dots, V_n) = P_H\, p(U_1, \dots, U_n)|_H$

$\rightsquigarrow \|p(V_1, \dots, V_n)\| \leq \|p(U_1, \dots, U_n)\|$

$C^*(\{U_1, \dots, U_n\}) \simeq C(X)$, $X \subseteq \mathbb{T}^n$ compact $\square$

Cor 5.3: $V_1, \dots, V_n$ commuting isometries

$(V_j^* V_i) \geq (V_i V_j^*) \geq 0$

Proof: $U_1, \dots, U_n$ as in 5.1, $K = H \oplus H^\perp$

$\rightsquigarrow U_i = \begin{pmatrix} V_i & X_i \\ 0 & Y_i \end{pmatrix}$ $(V_i H \subseteq H)$, $U_j^* U_i = U_i U_j^*$

$\overset{(1,1)}{\rightsquigarrow} (V_j^* V_i) = (V_i V_j^*) + (X_i X_j^*)$, $(V_i V_j^*) = (V_1^* \cdots V_n^*)^* (V_1^* \cdots V_n^*)$

$\square$

Thm 5.4: G abelian group, $P \subset G$ spanning cone ((i) $0 \in P$ (ii) $g_1, g_2 \in P \Rightarrow g_1 + g_2 \in P$

(iii) $g \in G \Rightarrow \exists g_1, g_2 \in P$ $g = g_1 - g_2$)

Let $\rho: P \to B(H)$ semigrouphomomorphism, s.t. $\rho(g)$ isometry for all $g \in P \Rightarrow \exists K \supseteq H$ Hilbert space, $\pi: G \to B(K)$ unitary represetation s.t. $\rho(g) = P_H \pi(g)|_H$ for all $g \in P$

Proof: $\Phi: G \to B(H)$, $g = g_1 - g_2 \mapsto \rho(g_2)^* \rho(g_1)$

Let $g = g_1 - g_2 = g_3 - g_4$

$\rightsquigarrow \rho(g_2)^* \rho(g_1) = \rho(g_2)^* \rho(g_3)^* \rho(g_3) \rho(g_1)$

$= \rho(g_2 + g_3)^* \rho(g_3 + g_1) = \rho(g_4 + g_1)^* \rho(g_4 + g_1) = \rho(g_4)^* \rho(g_3)$

Φ completely positive definite ($\Leftrightarrow g_1, \dots, g_n \in G$ $(\Phi(-g_i + g_j))$ positive)

$g_1, \dots, g_n$, $g_i = p_i - q_i$, $p_i, q_i \in P$, $1 \leq i \leq n$

$(\Phi(-g_i + g_j)) = (\rho(p_i + q_j)^* \rho(q_i + p_j)) = D^* (\underbrace{\rho(q_j)^*}_{V} \rho(q_i)) D \geq 0$

$D = \begin{pmatrix} \rho(p_1) & & \\ & \ddots & \\ & & \rho(p_n) \end{pmatrix} \xrightarrow{\text{Naimark}} \exists K \supseteq H$, $\pi: G \to B(K)$ unitary rep,

$V: H \to K$ $\Phi(g) = V^* \pi(g) V$, $1 = \Phi(0) = V^* V$,

$VH \simeq H$ $\square$

$C^*(\{U_1, \dots, U_n\}) \simeq C(X)$, $X \subseteq \mathbb{T}^n$ compact $\square$

Thm 5.5 (Ando): T_1, T_2 commuting contractions on Hilbert space H. Then $\exists\, \mathcal{U} \supseteq H$ Hilbert space, U_1, U_2 commuting unitaries on $\mathcal{U}$: $T_1^n T_2^m = P_H U_1^n U_2^m|_H$, $n, m \geq 0$

Proof: To show: $\exists$ commuting isometries V_1, V_2: $T_1^n T_2^m = P_H V_1^n V_2^m|_H$ $(*)$

Let $V_1: \ell^2(H) \to \ell^2(H)$, $(h_1, h_2, \dots) \mapsto (T_1 h_1, \underbrace{(1 - T_1^* T_1)^{1/2}}_{= D_1} h_1, 0, h_2, \dots)$

$V_1 V_2 (h_1, h_2, \dots) = (T_1 T_2 h_1, D_1 T_2 h_1, 0, D_2 h_1, 0, h_2, \dots)$

$V_2 V_1 (h_1, h_2, \dots) = (T_2 T_1 h_1, D_2 T_1 h_1, 0, D_1 h_1, 0, h_2, \dots)$

Assume $\exists\, U: H^{(4)} \to H^{(4)}$, $U((D_1 T_2 h, 0, D_2 h, 0)) = (D_2 T_1 h, 0, D_1 h, 0)$

Let $W: \ell^2(H) \to \ell^2(H)$, $(h_1, h_2, \dots) \mapsto (h_1, U(h_2, h_3, h_4, h_5), U(h_6, h_7, \dots), \dots)$

$\Rightarrow W V_1, V_2 W^{-1}$ commuting isometries, satisfy $(*)$

Existence of U: Isometry between $\{(D_1 T_2 h, 0, D_2 h, 0) : h \in H\} =: M_1$ and $\{(D_2 T_1 h, 0, D_1 h, 0) : h \in H\} =: M_2$

$\{(0, h, 0, 0)\}$

$\dim H = \dim H^{(4)} \geq \dim M_i^{\perp} \geq \dim H$ $\square$

Cor 5.6 T_1, T_2 commuting contractions on H, p_{ij}, $i, j = 1, \dots, n$ polynomials in two variables

$\Rightarrow \|(p_{ij}(T_1, T_2))\|_{B(H^{(n)})} \leq \sup\{\|(p_{ij}(z_1, z_2))\|_{M_n} : |z_i| \leq 1\}$

Thm 5.4: G abelian group, $P \subset G$ spanning cone)

$\dots \in P \Rightarrow g_1 + g_2 \in P$

Proof $\Phi: G \to B(H)$, $g = g_1 - g_2 \mapsto \rho(g_2)^* \rho(g_1)$

(Let $g = g_1 - g_2 = g_3 - g_4$ ...

Cor 5.8 (Commutant lifting theorem). Let T contraction on a Hilbert space, (U,K) minimal unitary dilation of T. R commutes with $T \Rightarrow \exists S$ commuting with U s.t. $\|R\| = \|S\|$

(A intertwines T_1 and $T_2 \Leftrightarrow AT_1 = T_2A$)

Cor 5.9 (Intertwining dilation theorem) T_1, T_2 contractions on Hilbert spaces H_i, minimal unitary

Example 5.7 $p(z_1,z_2,z_3) = z_1^2+z_2^2+z_3^2-2z_1z_3-2z_2z_3-2z_1z_2$

$$A_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & & & & \\ \sqrt{3} & & 0 & & \\ 0 & & & & \\ 0 & & & & \\ 0 & 1 & -1 & -1 & 0 \end{pmatrix}, \quad A_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & & & & \\ 0 & & 0 & & \\ \sqrt{3} & & & & \\ 0 & & & & \\ 0 & -1 & 1 & -1 & 0 \end{pmatrix},$$

$$A_4 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & & & & \\ 0 & & 0 & & \\ 0 & & & & \\ \sqrt{3} & & & & \\ 0 & -1 & -1 & 1 & 0 \end{pmatrix} \quad H = \mathbb{C}^5$$

$$\Rightarrow \|p\|_\infty = 5, \quad p(A_2,A_3,A_4) = \begin{pmatrix} 0 & & & & \\ 0 & & 0 & & \\ 0 & & & & \\ 0 & & & & \\ 3\sqrt{3} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$3\sqrt{3} > 5$ ↯

Not known $\|p(T_1,\dots,T_n)\| \le C_n \|p\|_\infty$?

Similar $\|p(T_1,\dots,T_n)\| \le C_n\|p\|_\infty \;\forall T_1,\dots,T_n$

Universal operator algebra for n-tuples of comm contractions

$\mathcal{P}_n$ = polynomials in n variables, $\|p\|_u = \sup\{\|p(T_1,\dots,T_n)\| \mid T_1,\dots,T_n \text{ comm contr}\}$

Supremum achieved $\|p(T_{1,k},\dots,T_{n,k})\| \to \|p\|_u$, $T_i^{(p)} = \bigoplus_k T_{i,k}$

For $p \in \mathcal{P}_n$ choose $(T_1^{(p)},\dots,T_n^{(p)})$, set $T_i = \bigoplus_p T_i^{(p)}$ on H

$\|p\|_u = \|p(T_1,\dots,T_n)\|$, $\pi: \mathcal{P}_n \to \mathcal{B}(H)$, $p \mapsto p(T_1,\dots,T_n)$

isometric homomorphism

$\|(p_{ij})\|_{u,k} = \sup\|(p_{ij}(T_1,\dots,T_n))\| \;\forall k \rightsquigarrow \pi$ completely isometric

$(\mathcal{P}_n, \|\;\|_{u,k})$ universal operator algebra

Cor 5.8 (Commutant lifting theorem). Let T contraction on a Hilbert space, $(U, \mathcal{K})$ minimal unitary dilation of T. R commutes with $T \Rightarrow \exists\, S$ commuting with U s.t. $\|R\| = \|S\|$ and $RT^n = P_H S U^n|_H$

Proof Scaling $\rightsquigarrow \|R\| = 1 \overset{\text{Ando}}{\Longrightarrow} T^n R^m = P_H U_1^n U_2^m|_H$, U_1, U_2 commuting unitaries on $\mathcal{K}_1$

$(U_1 \mathcal{K} \subseteq \mathcal{K}, U_1 \mathcal{K}^\perp \subseteq \mathcal{K}^\perp)$

$(U, \mathcal{K})$ minimal $\mathcal{K} = \overline{\{U_1^n H,\ n \in \mathbb{Z}\}}$ reducing subspace for U_1, $U = P_{\mathcal{K}} U_1|_{\mathcal{K}}$

$\mathcal{K}_1 = \mathcal{K} \oplus \mathcal{K}^\perp$ $\quad U_1 = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, $U_2 = \begin{pmatrix} S & B \\ C & D \end{pmatrix}$ commute $\overset{(1,1)}{\Longrightarrow}$ U, S commute

$\Rightarrow P_{\mathcal{K}} U_2 U_1^n|_{\mathcal{K}} = SU^n \quad P_H S U^n|_H = RT^n|_H$, $1 = \|R\| \le \|S\| \le \|U_2\| = 1$ $\square$

A intertwines T_1 and $T_2 \Leftrightarrow AT_1 = T_2 A$

Cor 5.9 (Intertwining dilation theorem) T_1, T_2 contractions on Hilbert spaces H_i, minimal unitary dilations $(U_i, \mathcal{K}_i)$. If A intertwines T_1 and $T_2 \Rightarrow \exists\, R$ intertwining U_1 and U_2 s.t. $\|A\| = \|R\|$ and

$$AT_1^n = T_2^n A = P_{H_2} R U_1^n|_{H_1} = P_{H_2} U_2^n|_{H_1}, \quad n \ge 0$$

Proof: $AT_1 = T_2 A \Leftrightarrow \begin{pmatrix} 0 & 0 \\ \ldots \end{pmatrix}$ … commutes with $\begin{pmatrix} T_1 & 0 \\ \ldots \end{pmatrix} = \hat{T}$ | Motivation $L^2(\mathbb{T}, H) \simeq \ell^2_{\mathbb{Z}}(H)$ via formal Fourier Serie…

<u>Proof</u> $AT_1 = T_2 A \iff \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \hat{A}$ commutes with $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} = \hat{T}$

$\hat{U} = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$ on $\mathcal{K}_1 \oplus \mathcal{K}_2$ minimal unitary dilation of $\hat{T}$

$\overset{5.8}{\rightsquigarrow} \hat{A}\hat{T}^n = P_{H_1 \oplus H_2} \hat{R} \hat{U}^n|_{H_1 \oplus H_2}$ with $\hat{R} = \begin{pmatrix} \tilde{A} & B \\ R & D \end{pmatrix}$

$\overset{(2,1)}{\rightsquigarrow} AT^n = P_{H_2} R U_1^n|_{H_1}, \; n \geq 0$

$\|A\| \leq \|R\| \leq \|\hat{R}\| = \|\hat{A}\| = \|A\| \; \square$

<u>Motivation</u> $L^2(\mathbb{T}, H) \simeq \ell^2_{\mathbb{Z}}(H)$ via formal Fourier series

$f \in L^2(\mathbb{T}, H): \sum_{n=-\infty}^{\infty} e^{in\theta} h_n, \; h_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta}) d\theta$

$\|f\|^2 = \sum_{n=-\infty}^{\infty} \|h_n\|^2$

$B_r: \mathbb{T} \to \mathcal{B}(H), \; e^{i\theta} \mapsto \sum_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\theta}, \; A_n \in \mathcal{B}(H), \; 0<r<1$ bounded uniformly

$M_{B_r}: L^2(\mathbb{T}, H) \to L^2(\mathbb{T}, H), \; f \mapsto B_r f$ bounded $\|M_{B_r}\| = \sup_\theta \|B_r(e^{i\theta})\|$

Using $L^2(\mathbb{T}, H) \simeq \ell^2_{\mathbb{Z}}(H)$

$$M_{B_r} \hat{=} \left(r^{|i-j|} A_{i-j}\right)_{i,j=-\infty}^{\infty}, \; \|(A_{i-j})\| = \sup_{r<1} \|B_r\|$$

If $\|(A_{i-j})\| < \infty$, identify (A_{i-j}) with M_B, $B(e^{i\theta}) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}$

$(U, \mathcal{K})$ minimal $\mathcal{K} = \{U_1 H$ … reducing subspace for U_1, $U = P_{\mathcal{K}} U_1|_{\mathcal{K}}$

Theorem 5.10. H separable Hilbert space, $A_n \in \mathcal{B}(H)$, $n \geq 0$. Then $(A_{i+j})_{i,j=0}^{\infty}$ (Hankel Matrix) bounded on $\ell^2(H)$ $\Leftrightarrow$ $\exists A_n \in \mathcal{B}(H)$, $n<0$ such that for $B_r : e^{i\theta} \mapsto \sum_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\theta}$ $\sup_{r<1} \|B_r\|_\infty < \infty$

In this case $\exists A_n \in \mathcal{B}(H)$, $n<0$ $\|(A_{i+j})\| = \|B\|_\infty = \sup_{r<1} \|B_r\|_\infty$

Proof $S = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & 1 & \ddots \\ & 0 & \ddots \end{pmatrix}$ right shift on $\ell^2(H)$, $S^* = \begin{pmatrix} 0 & 1 & 0 & \\ & 0 & 1 & 0 \\ & & \ddots & \ddots \end{pmatrix}$ left shift

$\Rightarrow (A_{i+j})$ intertwines S, S^*. Minimal unitary dilations of S, S^* are $\hat{S}, \hat{S}^*$ left/right shifts on $\ell^2_{\mathbb{Z}}(H)$

$\Rightarrow$ $R = (R_{ij}) : \ell^2_{\mathbb{Z}}(H) \to \ell^2_{\mathbb{Z}}(H)$, $\|R\| = \|(A_{i+j})\|$, $\hat{S}^* R = R\hat{S}$

$\Rightarrow \exists R_n \in \mathcal{B}(H), n \in \mathbb{Z} : R_{i,j} = R_{i+j}$, $R_n = A_n$, $n \geq 0$

$W : \ell^2_{\mathbb{Z}}(H) \to \ell^2_{\mathbb{Z}}(H)$, $(\dots, h_{-1}, h_0, h_1, \dots) \mapsto (\dots, h_1, h_0, h_{-1}, \dots)$ unitary

$\rightsquigarrow RW = (R_{i-j})$, $\|(A_{i+j})\| = \|R\| = \|RW\| = \|\sum_{n=-\infty}^{\infty} R_n e^{in\theta}\|_\infty$

$\square$