Commuting contractions on Hilbert spaces

Thm (Dilation theorem for commuting isometries)
Let $\{V_1 \dots V_n\}$ be a set of commuting isometries on a Hilbert space $\mathcal{H}$.
Then there exists a Hilbert space $K \supseteq \mathcal{H}$ and a set of commuting unitaries $\{U_1 \dots U_n\}$ on K s.t. $V_1^{m_1} \dots V_n^{m_n} = P_{\mathcal{H}} V_1^{m_1} \dots U_n^{m_n}|_{\mathcal{H}}$ $\forall \{m_1 \dots m_n\} \in \mathbb{N}_0^n$.

Corollary Let $V_1 \dots V_n$ commuting isometries
p_{ij} polynomial in n-variables
$\Rightarrow \| p_{ij}(V_1 \dots V_n) \|_{B(H^n)} \leq \sup \{ \| p_{ij}(z_1 \dots z_n) \| \mid |z_k| \leq 1 \quad 1 \leq k \leq n \}$

Pf: Any poly in n-comm isometries can be written as the compression of a polynomial in n unitaries.

$$p(V_1 \dots V_n) = P_H \, p(U_1 \dots U_n)|_H$$

$C^*(U_1 \dots U_n) \simeq C(X)$ $\quad X \subseteq \mathbb{T}^n$ compact subset

Corollary 5.3 $V_1 \dots V_n$ commuting isometries
$(V_j^* V_i) \geq (V_i V_j^*) \geq 0$
Pf: take $U_1, \dots, U_n$ as in 5.1 $\quad K = H \oplus H^\perp$

$\rightarrow U_i = \begin{pmatrix} V_i & X_i \\ & Y_i \end{pmatrix}$ $\quad U_j^* U_i = U_i U_j^*$

$V_i H \subseteq H$

$\rightarrow (V_j^* V_i) = (V_i V_j^*) + X_i X_j^*$

rewrite $(V_i V_j^*) =$

$\square$

We can get a generalisation of the last thm.

Thm 5.4 Let G be an Abelian group. $P \subset G$ spanning cone, i.e.
1) $0 \in P$;
2) P is closed under addition $\quad g_1, g_2 \in P \Rightarrow g_1 + g_2 \in P$
3) $\forall g \in P \; \exists \; g_1, g_2$ s.t. $g = g_1 - g_2$ (spanning condition)

Let $p: P \to B(H)$ be a semigroup homomorphism s.t. $p(g)$ is an isometry $\forall g \in P$.
$\Rightarrow \exists K \supseteq H$ Hilbert space & $\pi: G \to B(K)$ a unitary rep. such that $p(g) = P_H \pi(g)|_H$ $\forall g \in P$

Proof: $\quad \phi: G \to B(H) \qquad g = g_1 - g_2 \mapsto p(g_2)^* p(g_1)$

$$g = g_1 - g_2 = g_3 - g_4$$

$$\Rightarrow p(g_2)^* p(g_1) = p(g_2)^* p(g_3)^* p(g_3) p(g_1) =$$
$$= p(g_2 + g_3)^* p(g_3 + g_1) =$$
$$= p(g_4 + g_1)^* p(g_3 + g_1) = p(g_4)^* p(g_3)$$

$\Rightarrow$ well-defined

Φ is completely positive: consider $g_1 \dots g_n$ s.t. $g_i = p_i - q_i \quad p_i, q_i \in P \quad 1 \leq i \leq n$

$\Rightarrow \Phi(-g_i + g_j) = p(p_i + q_j)^* p(q_i + p_j) = \underset{\text{diagonal}}{D^*} (p(q_j)^* p(q_i)) D$ $\quad$ positive $\quad \Rightarrow \Phi$ completely positive.

Naimark's dilation thm gives us $K \subseteq H$, $\pi : G \to B(K)$ unitary rep, $V : H \to K$ s.t $\phi(g) = V^* \pi(g) V$, $V^*V = \phi(0) = 1$

$\Rightarrow VH \cong H$ isometrically $\square$

let us move to contractions

Thm 5.5 (Ando's thm)

let T_1, T_2 commuting contractions on a Hilbert space H. Then $\exists K \supseteq H$, U_1, U_2 commuting unitaries on K. then

$$T_1^n T_2^m = P_H (U_1^n U_2^m)|_H \qquad n, m \geq 0$$

Proof: to show: $\exists$ commuting isometries V_1, V_2 s.t

$$T_1^n T_2^m = P_{\bar{K}} V_1^n V_2^m |_{\bar{K}}$$

let $V : \ell^2(\bar{K}) \to \ell^2(\bar{K})$ $\quad (h_1 \dots h_2) \mapsto (T_1 h_1, (1-T_1^*T_1)^{1/2}, 0, h_2 \dots$

NB: not commuting in general:

$$(V_1 V_2)(h_1, h_2, \dots) = (T_1 T_2 h, D_1 T_2 h_1, 0, D_2 h, 0, h_2 \dots)$$
$$(V_2 V_1)(h_1, h_2, \dots) = (T_2 T_1 h, D_2 T_1 h_1, 0, D_1 h_1, 0, h_2 \dots)$$

Assume that $\exists U : H^{(4)} \to H^{(4)}$

let $W : \ell^2(H) \to \ell^2(H)$ s.t $(h_1 \dots h_2) \mapsto (h_1,$

$\Rightarrow W V_1 V_2 W^{-1}$ are commuting isometries & they satisfy (*)

what is missing is existence of U.

isometry between $\{(D_1 T_2 h_1, 0, D_2 h_1, 0)\} = M_1$

$\{(D_2 T_1 h_1, 0, D_1 h_1, 0)\} = M_2$

$\dim H = \dim H^{(4)} \geq \dim M_i^{\perp} \geq \dim H$ $\square$

Corollary (von Neumann type inequality) let T_1, T_2 commuting contractions, p^i $i = 1 \dots n$ polynomials in two variables

$$\Rightarrow \| p^i(T_1, T_2) \|_{B(H^{(n)})} \leq \sup \{ \| p_{ij}(z_1, z_2) \|_{M_n} : |z_1| \leq 1, |z_2| \leq 1 \}$$

NB VN fails for 3 or more commuting contractions

Counterexample:

on $\mathbb{C}^5$, consider operators

$$A_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{pmatrix} \qquad A_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \end{pmatrix}$$

$$A_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \end{pmatrix}, \quad p(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3$$

Not known if $\|p(T_1 \dots T_n)\| \leq K_n \|p\|_\infty$

lower bounds: $K_n \geq \frac{\sqrt{n}}{11}$

$\Rightarrow$ let P_n denote the algebra of polynomials in n-variables

set $\|p\|_u = \sup \|p(T_1 \dots T_n)\|$ where sup is over the set of n-tuples of commuting contraction.

- for every p $\exists$ n-tuple where the supremum is achieved
- for each p choose commuting n-tuple where $\|p\|_u$ is attained

$\rightarrow$ form the direct sum of all such n-tuples

$\Rightarrow$ get a single n-tuple for every polynomial. Then

$\pi : P_n \to B(H)$
$p \mapsto p(T_1 \dots T_n)$

is an isometric homomorphism.

$\rightarrow$ universal op. algebra for n-tuples of comm. contractions

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Corollary (commutant lifting theorem)

Let T be a contraction on a Hilbert space, and let (U, K) be the minimal unitary dilation of T. If R commutes with T, then there exists S commuting with U s.t $\|R\| = \|S\|$ and

$$RT^n = P_H S U^n |_H$$

Def: Given $T_i \in B(H_i)$ $i = 1, 2$, $A \in B(H_1, H_2)$, we say that A intertwines T_1, T_2 provided $AT_1 = T_2 A$

<u>Corollary</u> (Intertwining dilation theorem) let T_i $i=1,2$ be contractions on H with minimal unitary dilations (U_i, K_i). If A intertwines T_1 & T_2 $\Rightarrow$ $\exists$ R intertwining U_1, U_2 s.t $\|A\| = \|R\|$ &

$$A T_1^n = T_2^n A = P_{H_2} R U_1^n|_{H_1} = P_{H_2} U_2^n R|_{H_1} \quad \forall n \geq 0.$$

<u>Pf</u>: we observe that the minimal unitary dilation of

$$\hat{T} = \begin{pmatrix} T_1 & \\ & T_2 \end{pmatrix} \quad \text{is} \quad \hat{U} = \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix} \quad \text{on} \quad K_1 \oplus K_2$$

we can apply commutant lifting to $\hat{A} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \Rightarrow \hat{R} = \begin{pmatrix} \hat{A} & B \\ R & D \end{pmatrix}$

satisfying $\hat{A}\hat{T}^n = P_{H_1 \oplus H_2} \hat{R} \hat{U}^n|_{H_1 \oplus H_2}$

component-wise $\Rightarrow$ $AT^n = P_{H_2} R U_1^n|_{H}$ $\quad n \geq 0$

moreover $\|A\| \leq \|R\| \leq \|\hat{R}\| = \|\hat{A}\| = \|A\| \Rightarrow \|A\| = \|R\|$ $\quad \square$

<u>Applications</u>: norms of operator valued matrices

$\mathcal{H}$ separable, $f: \mathbb{T} \to \mathcal{H}$ measurable & square integrable

let

$$h_n = \frac{1}{(2\pi)} \int e^{-in\theta} f(e^{i\theta})\, d\theta$$

$$\Rightarrow \sum e^{in\theta} h_n$$

$$L^2(\mathbb{T}, H) = L^2(\mathbb{T}) \otimes H = \ell^2(\mathbb{Z}) \otimes \mathcal{H} = \ell^2_{\mathbb{Z}}(H) \quad +$$

A_n a norm-bounded sequence of operators. $0 < r < 1$

$$B_r(e^{i\theta}) = \sum A_n r^{|n|} e^{in\theta}$$

$\uparrow$ converges

$\Downarrow$

continuous op. valued function $\mathbb{T} \to B(\mathcal{H})$.

M_{B_r} bounded as operator on $L^2(\mathbb{T}, H)$: $\|M_{B_r}\| = \|B_r\|_\infty$

$\Rightarrow$ using the above $+$: operator valued Toeplitz matrix.

Theorem let $\mathcal{H}$ be separable. $A_n \in \mathcal{B}(\mathcal{H})$ $n \geq 0$ a sequence of operators

The operator valued Hankel matrix $(A_{i+j})_{i,j=0}^{\infty}$ is bounded on $\ell^2(\mathcal{H}) \iff$

$\exists\, A_n \in \mathcal{B}(\mathcal{H})$ $n < 0$ s.t $\|B\|_\infty = \sup_{r<1} \|B_r\| < +\infty$

$$B(e^{i\theta}) = \sum_{n=-\infty}^{+\infty} A_n e^{in\theta}.$$

Moreover, there exist a particular choice of A_n $n<0$ s.t

$$\|A_{i+j}\| = \|B\|_\infty$$

Proof: relies on the intertwining dilation theorem.