Chapter 6 Completely positive maps into M_n

$\subseteq \mathcal{A}$ operator system $\quad \mathcal{M} \subseteq \mathcal{A}$ operator space

Thm (Arveson's ext. thm)

$\varphi : S \to B(H)$ is com. pos. then $\exists$

com. pos. $\psi : \mathcal{A} \longrightarrow B(H)$

$$\begin{array}{ccc} \phi & \longmapsto & s_\phi \\ \phi_s & \longleftarrow & s \\ \cap & & \cap \\ \mathrm{Hom}(\mathcal{M}, M_n) & & \mathrm{Hom}(M_n(\mathcal{M}), \mathbb{C}) \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{M}, M_n) & \simeq & \mathrm{Hom}(M_n(\mathcal{M}), \mathbb{C}) \\ \cong & & \cong \\ \mathrm{Hom}(\mathcal{M}, \mathrm{Hom}(M_n, \mathbb{C})) & \simeq & \mathrm{Hom}(M_n \otimes \mathcal{M}, \mathbb{C}) \end{array}$$

suffices to consider $(a_i^* a_j)$ $a_1, \dots, a_m \in A$ by lemma 3.13

$\langle \varphi_m((a_i^* a_j)) x, x \rangle$

take $x = x_1 \oplus \dots \oplus x_m$

$x_i = \sum_k \lambda_{ik} e_k$

Thm 6.1

$\phi : S \to M_m$ TFAE

(i) ϕ com pos.

(ii) ϕ m-pos.

(iii) s_ϕ pos.

Proof: (i) $\to$ (ii) $\to$ (iii) ✓

Assume s_ϕ pos.

extend s_ϕ to

$s : M_m(A) \to \mathbb{C}$

$\varphi := \phi_s$ extends ϕ will show φ is m-pos

$s_\phi((a_{ij})) = \sum_{ij} \phi(a_{ij})_{ij}$

$s_\phi((a_{ij})) = \langle \phi_m((a_{ij}))e, e \rangle$

$e = e_1 \oplus \dots \oplus e_m \in (\mathbb{C}^m)^m$

$\phi_s(a)_{ij} = s(a \otimes E_{ij})$

suffices to consider $(a_i^* a_j)$ $a_1, \dots, a_m \in A$ by lemma 3.13

$\langle \psi_m((a_i^* a_j))x, x\rangle = \sum_{i,j} \langle \psi(a_i^* a_j)x_j, x_i\rangle$

take $x = x_1 \oplus \dots \oplus x_m$

$x_i = \sum_k \lambda_{ik} e_k$

$= \sum_{i,j,k,l} \lambda_{jk} \overline{\lambda_{il}} \underline{\langle \psi(a_i^* a_j) e_k, e_l\rangle} = (*)$

$A_i := \begin{pmatrix} \lambda_{i1} & \cdots & \lambda_{im} \\ & 0 & \end{pmatrix}$ then $A_i^* A_j = \sum_{k,l} \lambda_{jk} \overline{\lambda_{il}} E_{lk}$

$(*) = \sum_{i,j} s(a_i^* a_j \otimes A_i^* A_j) = s\left(\left(\sum_i a_i \otimes A_i\right)^* \left(\sum_j a_j \otimes A_j\right)\right) \geq 0$

<u>Thm 6.1</u>

$\phi : S \to M_n$ TFAE

(i) ϕ com pos.

(ii) ϕ n-pos.

(iii) s_ϕ pos.

<u>Proof</u>: (i) $\to$ (ii) $\to$ (iii) ✓

Assume s_ϕ pos.

extend s_ϕ to

$s : M_n(A) \to \mathbb{C}$

$\psi := \phi_s$ extends

ϕ Will show ψ is n-pos

$s_\phi((a_{ij})) = \sum_{i,j} \phi(a_{ij})_{ij}$

$s_\phi((a_{ij})) = \langle \phi_n((a_{ij}))e, e\rangle$

$e = e_1 \oplus \dots \oplus e_n \in (\mathbb{C}^n)^n$

$\phi_s(a)_{ij} = s(a \otimes E_{ij})$

suffices to consider $(a_i^* a_j)$ $a_1, \dots, a_m \in \mathcal{A}$ by lemma 3.13

take $x = x_1 \oplus \dots \oplus x_m$

$x_i = \sum_k \lambda_{ik} e_k$

$\langle \varphi_m((a_i^* a_j))x, x\rangle = \sum_{ij} \langle \varphi(a_i^* a_j)x_j, x_i\rangle = (*)$

$= \sum \lambda \bar{\lambda} \langle \varphi(a_i^* a_j) e_k, e_l\rangle$

Thm 6.2

$\phi : S \to M_m$ com. pos

$\exists$ com. pos ext $\psi : \mathcal{A} \to M_m$

Thm 6.3

$\phi : \mathcal{M} \to M_m$

Assume $1 \in \mathcal{M}$ $\phi(1) = 1$

then TFAE

(i) ϕ com. contr.

(ii) ϕ m-contr.

(iii) $\frac{1}{m} s_\phi$ contr.

Proof: (i) $\to$ (ii) $\to$ (iii) ✓

Assume (iii)

$\frac{1}{m} s_\phi$ unital

$S = \mathcal{M} + \mathcal{M}^*$

$\frac{1}{m} s_\phi$ extends to pos.

$\frac{1}{m} \bar{s}_\phi : S \to M_m$

$s_\phi((a_{ij})) = \sum_{ij} \phi(a_{ij})_{ij}$

$s_\phi((a_{ij})) = \langle \phi_m((a_{ij}))e, e\rangle$

$e = e_1 \oplus \dots \oplus e_m \in (\mathbb{C}^m)^m$

$\phi_s(a)_{ij} = s(a \otimes E_{ij})$

$S^+ \otimes M_n^+ = \left\{ \sum_i a_i \otimes A_i \;\middle|\; a_i \in S^+,\ A_i \in M_n^+ \right\}$

$S^+ \otimes M_n^+ \subseteq M_n(S)^+$

$a \otimes A = (\sqrt{a} \otimes \sqrt{A})^2$

Lemma 6.5

$\phi : S \to M_n$ pos.

$\longleftrightarrow S_\phi$ is pos. $S^+ \otimes M_n^+$

Proof: $a \in S^+$, $Q = (\bar{\alpha}_i \alpha_j) \in M_n^+$

$x = \alpha_1 e_1 + \dots + \alpha_n e_n \in \mathbb{C}^n$

$S_\phi(a \otimes Q) = \sum_{ij} \phi(\bar{\alpha}_i \alpha_j a)_{ij}$

$= \sum_{ij} \bar{\alpha}_i \alpha_j \langle \phi(a) e_j, e_i \rangle = \langle \phi(a) x, x \rangle$ □

$S_\phi((a_{ij})) = \sum_{ij} \phi(a_{ij})_{ij}$

$S_\phi((a_{ij})) = \langle \phi_n((a_{ij})) e, e \rangle$

$e = e_1 \oplus \dots \oplus e_n \in (\mathbb{C}^n)^n$

$\phi_s(a)_{ij} = s(a \otimes E_{ij})$

$\psi := S_{S_\phi}$ ψ extends ϕ and is com pos. and unital

so ψ com. contr. $\to$ ϕ com contr. □

Thm 6.4

$\phi : \mathcal{M} \to M_n$ $1 \in \mathcal{M}$ $\phi(1) = 1$

is n-contr. then $\exists$ com pos.

extension $\psi : A \to M_n$

Lemma A

$\phi : S \to M_m$ pos. $(a_{ij}) \in M_n(S)$ s.t. $\phi_n((a_{ij})) \not\geq 0$ then $\exists$ $\phi' : S \to M_m$ pos. unital s.t. $\phi'_n((a_{ij})) \not\geq 0$

Thm 6.6

TFAE

(i) every positive $\phi : S \to M_n$ is com. pos.

(ii) every pos. unital ϕ is com. pos.

(iii) $S^+ \otimes M_n^+$ is dense in $M_n(S)^+$

Proof: (i) $\Leftrightarrow$ (ii)

(iii) $\to$ (i) ✓

Assume $S^+ \otimes M_n^+$ not dense $P \in M_n(S)^+ \setminus \overline{S^+ \otimes M_n^+}$

exist $s : M_n(S) \to \mathbb{C}$

$s(S^+ \otimes M_n^+) \geq 0$

but ϕ_s not

$$s_\phi((a_{ij})) = \sum_{ij} \phi(a_{ij})_{ij}$$

$$s_\phi((a_{ij})) = \langle \phi_n((a_{ij}))e, e \rangle$$

$$e = e_1 \oplus \cdots \oplus e_n \in (\mathbb{C}^n)^n$$

$$\phi_s(a)_{ij} = s(a \otimes E_{ij})$$

Proof. (ii) $\longrightarrow$ (i)

we may assume

$B = B(H)$ let

$\phi : S \longrightarrow B(H)$ pos.

take $(a_{ij}) \in M_n(S)^+$

let $x = x_1 \oplus \cdots \oplus x_n \in H^n$

we need to show

$\langle \phi_n((a_{ij}))x, x \rangle \geq 0$

Let $F = \mathrm{span}\{x_i\}$

let $P_F : H \longrightarrow F$

be the proj.

let $\psi : S \longrightarrow B(F)$

be the compression of

ϕ i.e. $\psi(a) = P_F^* \phi(a) P_F$

$B(F) \cong M_k$ where $k = \dim F$

so ψ com. pos.

$s_\phi((a_{ij})) = \sum_{ij} \phi(a_{ij})_{ij}$

$s_\phi((a_{ij})) = \langle \phi_n((a_{ij}))e, e \rangle$

$e = e_1 \oplus \cdots \oplus e_n \in (\mathbb{C}^n)^n$

$\phi_s(a)_{ij} = s(a \otimes E_{ij})$

$\langle \phi_n((a_{ij}))x, x \rangle = \sum_{ij} \langle \phi(a_{ij})x_j, x_i \rangle$

$= \sum_{ij} \langle \psi(a_{ij})x_j, x_i \rangle = \langle \psi_n((a_{ij}))x, x \rangle \geq 0$

Cor 6.7

TFAE

(i) $\forall$ operator systems B and $\phi : S \to B$ pos. we have ϕ com. pos.

(ii) $\forall n$ and $\phi : S \to M_n$ pos. $\to \phi$ com. pos.

(iii) $\forall n$ $S^+ \otimes M_n^+$ is dense in $M_n(S)^+$

Thm 6.6

TFAE

(i) every positive $\phi : S \to M_n$ is com. pos.

(ii) every pos. unital ϕ is com. pos.

(iii) $S^+ \otimes M_n^+$ is dense in $M_n(S)^+$

<u>Proof</u> (ii) $\to$ (i)

we may assume $B = B(H)$ let $\phi : S \to B(H)$ pos.

take $(a_{ij}) \in M_n(S)^+$

let $x = x_1 \oplus \dots \oplus x_n \in H^n$

need to show $\langle \phi_n((a_{ij}))x, x \rangle \geq 0$

Let $F = \operatorname{span}\{x_i\}$

let $P_F : H \to F$ be the proj.

let $\psi : S \to B(F)$ be the compression of ϕ i.e. $\psi(a) = P_F^* \phi(a) P_F$

$B(F) = M_k$ where $k = \dim F$

so ψ com. pos.

$$\langle \phi_n((a_{ij}))x, x \rangle = \sum_{ij} \langle \phi(a_{ij})x_j, x_i \rangle = \sum_{ij} \langle \psi(a_{ij})x_j, x_i \rangle = \langle \psi_n((a_{ij}))x, x \rangle \geq 0$$

$$S_\phi((a_{ij})) = \sum_{ij} \phi(a_{ij})_{ij}$$

$$S_\phi((a_{ij})) = \langle \phi_n((a_{ij}))e, e \rangle$$

$$e = e_1 \oplus \dots \oplus e_n \in (\mathbb{C}^n)^n$$

$$\phi_s(a)_{ij} = s(a \otimes E_{ij})$$

Def

$x \in S$ we say it is partitionable wrt S or S has a part. of unity for x if $\forall \varepsilon > 0$ $\exists p_i \in S^+$ $\lambda_i \in \mathbb{C}$ with $|\lambda_i| \leq \|x\|$ $\sum_i p_i \leq 1$ and

$\| x - \sum_i \lambda_i p_i \| < \varepsilon$

$S' \subseteq S$ subset we say S has part. of unity for S' if x is part. wrt S $\forall x \in S'$

Lemma 6.8

$x \in S$ $\|x\| \leq 1$ TFAE

(i) x is part. wrt S

(ii) $\forall$ pos ϕ with domain S $\|\phi(x)\| \leq \|\phi(1)\|$

(iii) $\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix}$ is in the closure of $S^+ \otimes M_2^+$

Proof (i) $\rightarrow$ (ii):

Let $\varepsilon > 0$ and let p_i, λ_i as in the def. then

$\sum p_i \leq 1 \rightarrow \sum \phi(p_i) \leq \phi(1)$

$\leq \|\phi(1)\| \cdot 1$ by lemma 7.3

this implies that

$\| \sum_i \lambda_i p_i \| \leq \|\phi(1)\|$

$\|\phi(x)\| \leq \|\phi(x - \sum_i \lambda_i p_i)\|$

$+ \|\sum \lambda_i \phi(p_i)\| \leq$

$\varepsilon \cdot \|\phi\| + \|\phi(1)\|$

let $\varepsilon \to 0$

$\|\phi(x)\| \leq \|\phi(1)\|$

$S_\phi((a_{ij})) = \sum_{ij} \phi(a_{ij})_{ij}$

$S_\phi((a_{ij})) = \langle \phi_m((a_{ij}))e, e \rangle$

$e = e_1 \oplus \cdots \oplus e_m \in (\mathbb{C}^m)^m$

$\phi_s(a)_{ij} = s(a \otimes E_{ij})$

(ii) $\longrightarrow$ (iii): Assume $\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix} \notin S^+ \otimes M_2^+$

let $s: M_2(S) \to \mathbb{C}$ be s.t.

$s\left(\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix}\right) < 0$ but $s(S^+ \otimes M_2^+) \geq 0$

$\phi = \phi_s$ $\quad \phi_2\left(\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix}\right) \ngeq 0$ now let

ϕ' be pos unital s.t.

$\phi'_2\left(\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & \phi'(x) \\ \phi'(x)^* & 1 \end{pmatrix} \ngeq 0$

so lemma 3.1 $\longrightarrow \|\phi'(x)\| > 1$

Lemma 6.8

$x \in S$ $\quad \|x\| \leq 1$ $\quad$ TFAE

(i) x is pure wrt S

(ii) $\forall$ pos ϕ with domain S $\|\phi(x)\| \leq \|\phi(1)\|$

(iii) $\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix}$ is in the closure of $S^+ \otimes M_2^+$

(iii) $\to$ (i): Let $\varepsilon > 0$

and pick $P_i \in S^+$ $\quad Q_i \in M_2^+$

s.t. $\left\|\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix} - \sum_i P_i \otimes Q_i\right\| < \varepsilon$

now $Q_i = \begin{pmatrix} r_i & \lambda_i \\ \bar{\lambda}_i & t_i \end{pmatrix}$ $\quad r_i, t_i \geq 0$

$\sum_i r_i P_i < 1 + \varepsilon$

$\sum_i t_i P_i < 1 + \varepsilon$

$\|x - \sum_i \lambda_i P_i\| < \varepsilon$

let $s_i := \frac{r_i + t_i}{2}$

then $\left|\frac{\lambda_i}{s_i}\right| \leq 1$

$\frac{\sum s_i P_i}{1+\varepsilon} < 1$ now

$\left\|x - \frac{\sum_i \frac{\lambda_i}{s_i} s_i P_i}{1+\varepsilon}\right\| \leq$

$\left\|\frac{x(1+\varepsilon) - \sum_i \frac{\lambda_i}{s_i} s_i P_i}{1+\varepsilon}\right\| \leq \frac{\varepsilon}{1+\varepsilon}\|x\| + \frac{1}{1+\varepsilon}\|x - \sum_i \lambda_i P_i\| < 2\varepsilon$

$\square$

$s_\phi((a_{ij})) = \sum_{ij} \phi(a_{ij})_{ij}$

$s_\phi((a_{ij})) = \langle \phi_n((a_{ij}))e, e\rangle$

$e = e_1 \oplus \cdots \oplus e_n \in (\mathbb{C}^n)^n$

$\phi_s(a)_{ij} = s(a \otimes E_{ij})$

Thm 6.9

$1 \in S' \subseteq S$ subset then TFAE

(i) every pos map with domain S has norm $\|\phi(1)\|$ on S'

(ii) S has a part. of unity for S'

Cor 6.10

$B \subseteq A$ a subalgebra $S = B + B^*$ then S has a part of unity for B

Lemma 6.9

$x \in S$ $\|x\| \leq 1$ TFAE

(i) x is part. wrt S

(ii) $\forall$ pos ϕ with domain S ...

(iii) $\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix}$ is in the closure ...

$S^+ \otimes M_2^+$

Proof: by Cor 2.8 every pos map ϕ with domain S satisfies $\|\phi(a)\| \leq \|\phi(1)\|$ $\forall a \in B$ with $\|a\| \leq 1$ $\square$

let $s_i := \frac{r_i + t_i}{2}$

then $\left|\frac{\lambda_i}{s_i}\right| \leq 1$

$\frac{\sum s_i p_i}{1+\varepsilon} < 1$ now

$$\left\| x - \frac{\sum_i \frac{\lambda_i}{s_i} s_i p_i}{1+\varepsilon} \right\| \leq$$

$$\left\| \frac{x(1+\varepsilon) - \sum_i \frac{\lambda_i}{s_i} s_i p_i}{1+\varepsilon} \right\| \leq \frac{\varepsilon}{1+\varepsilon}\|x\| + \frac{1}{1+\varepsilon}\|x - \sum_i ...$$

$S_\phi((a_{ij}$ ...

$S_\phi((a_{ij})) =$ ...

$e = e_1 \oplus \cdots$

$\phi_s(a)_{ij} =$ ...

Def

$A \subseteq C(X)$ subalgebra is uniform if $1 \in A$ and separates points and is hypo-Dirichlet if the closure S of $A + \overline{A}$ has finite codim in $C(X)$

Example 6.11

let $0 < R_1 < R_2$ and $\mathbb{A} = \{ z \in \mathbb{C} \mid |R_1| \leq |z| \leq |R_2| \}$

let $\mathcal{R}(\mathbb{A})$ be the algebra of rational functions on $\mathbb{A}$ let $\Gamma_j = \{ z \in \mathbb{C} \mid |z| = R_j \}$ $j = 1, 2$

$\mathcal{R}(\mathbb{A}) \subseteq C(\partial \mathbb{A})$ let S closure of $\mathcal{R}(\mathbb{A}) + \overline{\mathcal{R}(\mathbb{A})}$

then S has codim 1 in $C(\partial \mathbb{A})$

$$S_j(f) := \frac{1}{2\pi} \int_0^{2\pi} f(R_j e^{it})\, dt$$

let $f = \sum_{k=-n}^{n} a_k z^k$ Laurent Poly then $S_1(f) = S_2(f) = a_0$

Example 6.11

let $0 < R_1 < R_2$ and $\mathbb{A} = \{ z \in \mathbb{C} \mid |R_1| \le |z| \le |R_2| \}$

let $\mathcal{R}(\mathbb{A})$ be the algebra of rational functions on $\mathbb{A}$ let $\Gamma_j = \{ z \in \mathbb{C} \mid |z| = R_j \}$ $j = 1, 2$

$\mathcal{R}(\mathbb{A}) \subseteq C(\partial \mathbb{A})$ let S closure of $\mathcal{R}(\mathbb{A}) + \overline{\mathcal{R}(\mathbb{A})}$

then S has codim 1 in $C(\partial \mathbb{A})$

$$S_j(f) := \frac{1}{2\pi} \int_0^{2\pi} f(R_j e^{it})\, dt$$

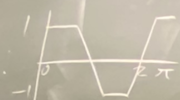
let $f = \sum_{k=-n}^{n} a_k z^k$ Laurent Poly then $S_1(f) = S_2(f) = a_0$

Since Laurent Poly's dense in $\mathcal{R}(\mathbb{A})$ and S_j self adjoint so $S_1(f) = S_2(f)$ $\forall f \in S$ so

$$S = \{ f \in C(\partial \mathbb{A}) \mid S_1(f) = S_2(f) \}$$

S does not have a part. of unity for S

let $f \in \underline{\underline{S}}$

f:  on Γ_1

multiply by i on Γ_2

by an exercise this implies $\exists\, \phi : S \to M_2$ pos. unital which is not contractive

(i) ϕ not contr.

(ii) ϕ not com. pos.

(iii) no pos. ext. to $C(\partial \mathbb{A})$

(iv) $\psi := \phi|_{\mathcal{R}(\mathbb{A})}$ is contr. but pos. extension

$\overline{\psi} = \phi : S \to M_2$ not contr.

(v) ψ has no contr. ext to S

(vi) ψ is not 2-contr.