Chapter 6 Completely positive maps into M_n.

$S \subseteq A$ op-system, $\mathcal{X} \subseteq A$ operator space.

Thm: (Arveson's extension theorem)

$\phi: S \to B(\mathcal{H})$ is completely positive. then $\exists$ completely positive $\psi: A \to B(\mathcal{H})$

- $S_\phi(a_{ij}) = \sum_{ij} \phi(a_{ij})_{ij}$. At the same time

$S_\phi(a_{ij}) = \langle \phi_n(a_{ij}) e, e \rangle$ $\qquad e = e_1 \oplus e_2 \oplus \dots \oplus e_n \in (\mathbb{C}^n)^n$

$\phi_s(a_{ij}) = s(a \otimes E_{ij})$

Theorem let $\phi: S \to M_n$. TFAE

1. ϕ is completely positive
2. ϕ is n-positive
3. S_ϕ is positive

Proof

$1 \Rightarrow 2$ definition

$2 \Rightarrow 3$ follows from the inner product formula.

$3 \Rightarrow 2$ Assume S_ϕ positive. Extend S_ϕ to $\underbrace{s: M_n(A) \to \mathbb{C}}_{\text{positive linear functional because of the above}}$.

$\Rightarrow$ S can be extended to $\psi := \phi_s$ extending ϕ.

Let us pick an arbitrary n & show that ψ is n-positive

It suffices to consider a positive matrix $(a_i^* a_j)$ $a_i \in A$ (C^*-alg) by lemma 3.13.

Compute $\langle \psi_m(a_i^* a_j) x, x \rangle = \sum_{ij} \langle \psi(a_i^* a_j) x_j, x_i \rangle =$

$x = x_1 \oplus \dots \oplus x_m \qquad x_i = \sum_k \lambda_{ik} e_k$

$$= \sum_{ijkl} \lambda_{jk} \overline{\lambda_{il}} \langle \psi(a_i^* a_j) e_k, e_l \rangle$$

If we now consider the $n \times n$ matrix $\begin{pmatrix} \lambda_{i1} & \dots & \lambda_{in} \\ & 0 & \end{pmatrix}$

then $A_i^* A_j = \sum_{k,l} \lambda_{jk} \overline{\lambda_{il}} E_{lk}$

Thm 6.2 $\phi : S \to M_n$ completely positive, $S \subseteq A$. $\exists$ completely positive extension $\psi : A \to M_n$

Thm 6.3 $\phi : \mathcal{M} \to M_n$. Assume $1 \in \mathcal{M}$, $\phi(1) = 1$.

Then TFAE
1) ϕ completely contractive
2) ϕ n-contractive
3) $\frac{1}{n} S\phi$ contractive

Proof $1 \Rightarrow 2 \Rightarrow 3$ ✓

$3 \Rightarrow 1$ $\quad \frac{1}{n} S\phi$ is not only contractive but also unital

contractive + unital $\Rightarrow$ positive on $\mathcal{M} + \mathcal{M}^* = S \longrightarrow M_n$, extends to $\overline{S\phi}$

Define ψ the linear fct. associated to $\overline{S\phi}$.

ψ extends ϕ

it is completely positive & unital

$\Rightarrow \psi$ is completely contractive $\Rightarrow \phi$ com. contr. $\square$

NB: we have actually proven more:

Thm $\varphi : \mathcal{M} \to M_n$, $1 \in \mathcal{M}$, $\phi(1) = 1$. If φ is n-contractive $\Rightarrow$ $\exists$ completely positive extension $\psi : A \to M$

(combining the previous results)

Consider $S^+ \otimes M_n^+ = \{ \sum a_i \otimes A_i \mid a_i \in S^+, A_i \in M_n^+ \}$

$S^+ \otimes M_n^+ \subseteq M_n(S)^+$

Lemma 6.5 $\phi : S \to M_n$ positive $\iff$ $S\phi$ is positive on $S^+ \otimes M_n^+$

Proof: let $a \in S^+$, $(\bar{\alpha}_i \alpha_j) \in M_n^+$ $\quad x = \alpha_1 e_1 + \dots + \alpha_n e_n \in \mathbb{C}^n$

$Q = \{ (\bar{\alpha}_i \alpha_j) \in M_n^+ \}$ $\quad S\phi(a \otimes Q) = \sum_{ij} \phi(\bar{\alpha}_i \alpha_j a)_{ij}$

$= \sum_{ij} \bar{\alpha}_i \alpha_j \langle \phi(a) e_j, e_i \rangle = \langle \phi(a) x, x \rangle$ $\square$

Lemma let $\phi : S \to M_n$ positive, $(a_{ij}) \in M_n(S)$ s.t. $\phi_m(a_{ij}) \not\geq 0$, then $\exists$ $\phi' : S \to M_n$ positive unital s.t. ϕ_m assumes

Thm 6.6 TFAE
1) every positive $\phi : S \to M_n$ is comp. positive
2) every unital, positive map $\phi : S \to M_n$ is completely positive
3) $S^+ \otimes M_n^+$ is dense in $M_n(S)^+$

Proof we already proved $1 \Leftrightarrow 2$ & $3 \Rightarrow 1$

Assume $S^+ \otimes M_n^+$ not dense

let $p \in M_n(S)^+$ not in the closure of $S^+ \otimes M_n^+$

By Krein-Milman $\exists$ s linear fct on $M_n(S)$, positive on $S^+ \otimes M_n^+$ but $s(p) < 0$.

Then the associated linear map $\phi_s : S \to M$ is positive but not completely positive (ϕ_n not positive) $\square$

Corollary 6.7 TFAE

1) $\forall$ operator system B & $\phi : S \to B$ positive ϕ is completely positive

2) for every n, every positive $\phi : S \to M_n$ comp. positive

3) $S^+ \otimes M_n^+$ dense in $M_n(S)^+$ $\forall n$

2) $\Rightarrow$ 3) follows from 6.6

1) holds $\Rightarrow$ 2)

to see 2) $\Rightarrow$ 1), assume $B = B(H)$. Given (a_{ij}) in $M_n(S^+)$

to check that $\phi_n((a_{ij}))$ enough to chose $x_1 \dots x_n \in H$ & check

$$\sum \langle \phi(a_{ij}) x_j, x_i \rangle \geq 0$$

F be f.dim. subspace spanned by $x_1 \dots x_n$

$\psi : S \to B(F)$ the compression

$F \simeq M_k$ $k = \dim(F)$ $\Rightarrow$ ψ completely positive by ii) & hence

$$0 \leq \sum \langle \psi(a_{ij}) x_j, x_i \rangle = \sum \langle \phi(a_{ij}) x_j, x_i \rangle$$ $\square$

Def S operator system. S has a partition of unity for x, x is partitionable wrt S, provided $\forall \varepsilon > 0$ $\exists$ $p_1 \dots p_n \in S$

$\sum p_i \leq 1$, scalars $\lambda_1 \dots \lambda_n$ with $|\lambda_i| \leq \|x\|$

such that $\|x - \sum \lambda_i p_i\| < \varepsilon$

We say that S has a partition of unity for H, provided every element of H is partitionable wrt S.

lemma 6.8 let S be an operator system, $x \in S$ $\|x\| \le 1$.

TFAE

1) x partitionable wrt S;

2) every positive map ϕ with domain S satisfies
$\|\phi(x)\| \le \|\phi(1)\|$

3) $\begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix}$ is in the closure of $S^+ \otimes M_2^+$

1) => 2) similar argument of thm 2.4. (using lemma 2.3)
to get $\|\phi(x)\| \le \|\phi(1)\|$

2) => 3) Assume 3) does not hold, that is $\exists\, s : M_2(S) \to \mathbb{C}$
s.t $s\left(\begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix}\right) < 0$ but $s(S^+ \otimes M_2(\mathbb{C})) \ge 0$

let ϕ be the lin. map associated to s

$\exists\, \phi'$ from Φ s.t $\phi_2'\left(\begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix}\right)$ not positive

By lemma 3.1 $\|\phi'(x)\| > 1 = \|\phi'(1)\|$ $\lightning$

3) => 1)
let $\epsilon > 0$ $p_i \in S^+$ $Q_i \in M_2^+$ $i = 1 \dots n$ s.t

$$\left\| \begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} - \sum p_i \otimes Q_i \right\| < \epsilon$$

now $Q_i = \begin{bmatrix} r_i & \lambda_i \\ \bar{\lambda}_i & t_i \end{bmatrix}$ with $r_i \ge 0$ $t_i \ge 0$ $\forall i$

$\sum r_i p_i < 1 + \epsilon$
$\sum t_i p_i < 1 + \epsilon$
$\|x - \sum \lambda_i p_i\| < \epsilon$

let $s_i = (r_i + t_i)/2$. then $|\lambda_i / s_i| < 1$

We have $1 - \epsilon < \sum s_i p_i < 1 + \epsilon$ &
$\|x - \sum (\lambda_i / s_i)\, s_i p_i\| < \epsilon$
$\Big\}$ x is partitionable $\square$

Thm 6.9 $\mathcal{K} \subseteq S$ $\ni 1$ S operator system. Every positive map on S has norm $\|\phi(1)\|$ when restricted to $\mathcal{K}$ $\iff$ S has a partition of unity for $\mathcal{K}$.

Corollary B C^* algebra, $1 \in B$, $A \subseteq B$ $1 \in A$ subalgebra,

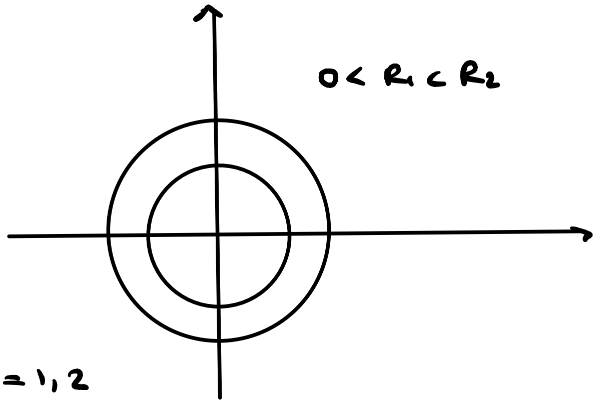
$S = A + A^*$ $\Rightarrow$ S has a partition of unity for A.

EXAMPLE illustrate the advantages of working with p of U.

Def $A \subseteq C(X)$ is *uniform* if $1 \in A$ and separates points.

A hypo-Dirichlet algebra is a uniform algebra $A \subseteq C(X)$ s.t. the closure of $A + \bar{A}$ has finite codimension in A.

EXAMPLE Annulus



$0 < R_1 < R_2$

$A = \{ z \in \mathbb{C} \mid R_1 < |z| < R_2 \}$

$R(A) :=$ algebra of rational functionals on A

$\Gamma_j = \{ z \in \mathbb{C} \mid |z| = R_j \}$ $j = 1, 2$

$R(A) \subseteq C(\partial A)$.

let $S :=$ closure of $R(A) + \overline{R(A)}$ in $C(\partial A)$

S has codimension 1 (due to Walsh)

Define functionals $s_j : C(\partial A) \to \mathbb{C}$ by

$$s_j(f) = \frac{1}{2\pi} \int_0^{2\pi} f(R_j e^{it})\, dt$$

Fact Laurent polynomials are dense in $R(A)$

for $f = \sum_{k=-n}^{+n} a_k z^k$ a finite Laurent poly, we have

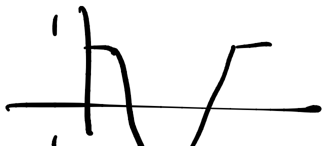
$$s_1(f) = s_2(f)$$

$$\Rightarrow \quad s_1(f) = s_2(f) \quad \forall f \in S$$

$$\Rightarrow \quad S = \{ f \in C(\partial A) \mid s_1(f) = s_2(f) \}$$

<u>claim</u> S has no partition of unity for S itself

let $f \in S$



on Γ_1, multiply by i on Γ_2

$s_1(f) = s_2(f)$

then f has no partition of unity in S.

<u>Consequences</u> : $\exists$ positive unital map $\phi : S \to M_2$ s.t

1) ϕ not contractive ;
2) ϕ not completely positive (otherwise completely contractive)
3) ϕ has no positive extension to $C(\partial A)$
   (positive on comm. C^* alg $\Rightarrow$ comp. pos)

4) $\varphi = \phi|_{R(A)}$ is a unital contraction but its positive extension $\overline{\varphi} = \phi$ is not .

5) φ has no contractive extension to S.

6) φ is not 2- contractive