Ch 7: Arveson's extension thm . i.e. ($B(\mathcal{H})$ is <u>injective</u>)

Lemma 7.1: Let X, Y be Banach spaces.

$B(X, Y^*)$ is isometrically isomorphic to Z^*;

where $Z = \overline{\text{span}\{x \otimes y\}}$. for $x \in X$. $y \in Y$

$x \otimes y \in X \otimes Y$.

$\text{span}\{x \otimes y\} \subseteq B(X, Y^*)^*$

$$x \otimes y : B(X, Y^*) \to \mathbb{C}$$
$$L \mapsto L(x)(y)$$

Ch 7: Arveson's extension thm ... i.e. ($B(\mathcal{H})$ is <u>injective</u>)

Lemma 7.1: Let X, Y be Banach spaces.

$B(X, Y^*)$ is isometrically isomorphic to Z^*,

where $Z = \overline{\text{span}\{x \otimes y\}}$

Pf: $\langle L, x \otimes y \rangle = L(x)(y)$

$B(X, Y^*) \hookrightarrow Z^*$

Only need to show for any $f \in Z^*$, we can construct an $L \in B(X, Y^*)$,

$f : Z \to \mathbb{C}$

Let $L(x) := f_x$, where

$f_x(y) := f(x \otimes y)$

$L(x)(y) = f(x \otimes y)$

$|L(x)(y)| \leq \|f\| \|x\| \|y\|$

thus L is bdd.

Def: We call the weak* top on $B(X, Y^*)$ the bounded weak (BW) top.

Lemma 7.2: Let $\{L_\lambda\}$ bdd net in $B(X, Y^*)$, then

$L_\lambda \to L$ in the sense of BW top

$\iff L_\lambda(x) \to L(x)$ in the sense of weak top for every $x \in X$.

Pf: $L_\lambda \to L \iff$ for every $x \otimes y \in X \otimes Y$, $L_\lambda(x \otimes y) = L_\lambda(x)(y) \to L(x \otimes y) = L(x)(y)$

Ch 7: Arveson's extension thm i.e. ($B(\mathcal{H})$ is injective)

In particular. $B(\mathcal{H})$ is iso to $B_1(\mathcal{H})^*$, where $B_1(\mathcal{H})$ is the set of trace-class oprs.

Prop 7.3: Let X be a Banach space. $\mathcal{H}$ be a Hilbert space.
then a bdd net $\{L_\lambda\}$ in $B(X, B(\mathcal{H}))$ is convergent to $L \in B(X, B(\mathcal{H}))$

$\Longleftrightarrow \langle L_\lambda(x) h, k\rangle \to \langle L(x) h, k\rangle, \quad h, k \in \mathcal{H}$

$R_{h,k}: \mathcal{H} \to \mathcal{H}$.

$v \mapsto \langle v, k\rangle h$

$\{R_{h,k}\}$ is dense in $B_1(\mathcal{H})$.

Thm 7.4: Let $\mathcal{A}$ be a C^*-alg.

$S \subseteq \mathcal{A}$ be a closed opr. sys.

Then $CP_r(S, \mathcal{H})$ is cmpt in the BW top.

Pf: $CP_r(S, \mathcal{H}) \subseteq B(S, B(\mathcal{H}))$, according to Banach - Alaoglu thm

we only need to show $CP_r(S, \mathcal{H})$ is a closed bounded subset.

Let $\{L_\lambda\}$ be a bdd net in $CP_r(S, \mathcal{H})$ s.t. $L_\lambda \to L$ in the sense of BW-top

Let $\mathcal{A}$ be a C^*-alg.

$S \subseteq \mathcal{A}$ be an opr. sys.

$CP_r(S, \mathcal{H}) = \{L \in B(S, B(\mathcal{H})) :$

L is completely positive.

$\|L\| \le r\}$

$\| = $

$\|L\|_{cb}$

Ch 7: Arveson's extension thm. i.e. ($B(\mathcal{H})$ is injective)

Thm 7.4: Let $\mathcal{A}$ be a C^*-alg.

$S \subseteq \mathcal{A}$ be a closed opr. sys.

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$(a_{i,j}) \in M_n(S)$. $\quad x = x_1 \oplus x_2 \oplus \cdots \oplus x_n \in \mathcal{H}^{\oplus n}$.

$\qquad\qquad\qquad\quad y = y_1 \oplus y_2 \oplus \cdots \oplus y_n$

$\langle L_\lambda(a_{i,j})x, y\rangle \longrightarrow \langle L(a_{i,j})x, y\rangle$ for every $x, y \in \mathcal{H}^{\oplus n}$

$\|L_\lambda\| \leq r$. $\quad \|L(a_{i,j})\| \leq r\|(a_{i,j})\| \Rightarrow \|L\|_{cb} \leq r$.

Ch 7: Arveson's extension thm. i.e. ($B(\mathcal{H})$ is <u>injective</u>)

Thm 7.5 (Arveson's extension theorm):

Let $\mathcal{A}$ be a C^*-alg. $S \subseteq \mathcal{A}$ be an opr system

if $\varphi: S \to B(\mathcal{H})$ is completely positive.

then there is a map $\psi: \mathcal{A} \to B(\mathcal{H})$. ψ is completely positive. $\psi|_S = \varphi$.

Pf: Let F be a finite dim subspace of $\mathcal{H}$.

def $\varphi_F : S \to \underline{B(F)}$

$x \mapsto P_F \varphi(x)|_F$

extent φ_F to $\psi_F : \mathcal{A} \to B(F)$ C.P.

$\psi_F' : \mathcal{A} \to \underline{B(\mathcal{H})}$

$a \mapsto \psi_F(a)$ on F

0 on $F^\perp$

Let $\mathcal{A}$ be a C^*-alg.

$S \subseteq \mathcal{A}$ be an opr. sys.

$\boxed{CP_r(S, \mathcal{H})} = \{ L \in B(S, B(\mathcal{H})) :$

L is completely positive.

$\|L\| \le r \}$

$\|L\|_{cb}$

w.r.t BW-top

Pf: Let F be a finite dim subspace of $\mathcal{H}$. Let A be a C^*-alg.

Ch 7: Arveson's extension thm. i.e. ($B(\mathcal{H})$ is injective)

$\{\psi_F'\}$ is a bdd net in $CP_r(S, \mathcal{H})$ with $r = \|\varphi\|$

For every $\tilde{F}$, we can find some F' s.t. $\tilde{F} \subseteq F'$

there exist a $\psi \in CP_r(S, \mathcal{H})$, s.t.

some subnet of $\{\psi_F'\}$ converges to ψ.

$$F' \supseteq F$$

Only need to show: $\psi|_S = \varphi$.

Let $a \in S$, $x, y \in \mathcal{H}$. Let $\boxed{F = \mathrm{span}\{x, y\}}$.

we can find $F' \supseteq F$,

$$\langle \psi(a)x, y\rangle = \langle \varphi(a)x, y\rangle = \langle \psi_{F'}'(a)x, y\rangle.$$

the set of F' is $\boxed{\text{cofinal}}$

Ch 7: Arveson's extension thm. i.e. ($B(\mathcal{H})$ is <u>injective</u>)

Corollary 7.6 (Arveson). Let $\mathcal{A}$ be a C^*-alg.

$\mathcal{M}$ be a subspace of $\mathcal{A}$ with $1 \subseteq \mathcal{M}$

$\varphi : \mathcal{M} \to B(\mathcal{H})$ be a unital. completely contraction.

Then there is a completely positive map $\psi : \mathcal{A} \to B(\mathcal{H})$

s.t $\psi|_{\mathcal{M}} = \varphi$.

Pf. Since φ is unital.

completely contr.

$\tilde{\varphi} : \mathcal{M} + \mathcal{M}^* \to B(\mathcal{H})$

$\tilde{\varphi}$ is completely positive.

Pf: Let F be a finite dim subspace of $\mathcal{H}$.

def $\varphi_F : S \to \underline{B(F)}$

$\longmapsto P_F\,\varphi(x)|$

Let $\mathcal{A}$ be a C^*-alg.

$S \subseteq \mathcal{A}$ be an opr. sys.

$CP_r(S, \mathcal{H}) = \{\, \in B(S, B(\mathcal{H}))$

Ch 7: Arveson's extension thm i.e. ($B(\mathcal{H})$ is injective)

CP 15

Def: A C^*-alg B is called injective if for every C^*-alg $\mathcal{A}$ and $S \subseteq \mathcal{A}$ be an opr. sys. every C.P. map $\varphi : S \to B$ can be extended to a C.P map $\psi : \mathcal{A} \to B$.

Ex 7.5 B is injective iff $B \subseteq B(H)$.
there is a C.P. $\varphi : B(H) \to B$. s.t.
$\varphi(b) = b$ for every $b \in B$.

Example injective O.S.

The set of circulant matrices is an injective O.S.

let $U = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & & 0 \end{pmatrix}_{n \times n} \in M_n(\mathbb{C})$

$C = \operatorname{span}\{Id, U, U^2 \ldots U^{n-1}\}$

Let A be a C^*-alg.
$S \subseteq A$ be an opr sys.

Let $\varphi : M_n(\mathbb{C}) \to C$

$$\varphi(X) = \frac{1}{n} \sum_{k=1}^{n} U^{*k} X \cdot U^{k}$$

$\varphi(c) = c$ for every $c \in C$.

$$\varphi : \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \frac{1}{3} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

✓

Ex 7.5 $\mathcal{B}$ is injective iff $\mathcal{B} \subseteq B(\mathcal{H})$.

Let $\mathcal{A}$ be a C^*-alg.

Ch 7: Arveson's extension thm i.e. ($B(\mathcal{H})$ is injective)

Chapter 15.

Def: ($\mathcal{B}$-dilation): Let $\mathcal{B}$ be a unital C^*-alg.

$\mathcal{A} \subseteq \mathcal{B}$ be a subalgebra.

we call $\mathcal{A}$ an operator alg.

A unital homomorphism $\varphi: \mathcal{A} \to B(\mathcal{H})$ is said to have a $\mathcal{B}$-dilation if there is a H.S. $\mathcal{K}$ s.t. $\mathcal{H} \subseteq \mathcal{K}$ and there is a $*$-hom $\pi: \mathcal{B} \to B(\mathcal{K})$ s.t.

$$\varphi(a) = P_{\mathcal{H}} \pi(a)\big|_{\mathcal{H}} \quad \text{for all } a \in \mathcal{A}.$$

Corollary 7.7 (Arveson). Let $\mathcal{B}$ be a C^*-alg.

$\mathcal{A} \subseteq \mathcal{B}$ be an opr. alg.

Let $\rho : \mathcal{A} \to B(\mathcal{H})$ be a unital homomorphism.

Let $\tilde{\rho} : \mathcal{A} + \mathcal{A}^* \to B(\mathcal{H})$ be the positive ext. of ρ

$$a \mapsto \rho(a)$$
$$a^* \mapsto \rho(a)^*$$

Then TFAE:

1) ρ has a $\mathcal{B}$-dilation

2) ρ is completely contract

3) $\tilde{\rho}$ is completely positive.

Pf: If ρ has a $\mathcal{B}$-dilation.

There is $\varphi : \mathcal{B} \to B(\mathcal{H})$

$$\varphi(b) = P_{\mathcal{H}} \pi(b)|_{\mathcal{H}}$$

Ch 7: Arveson's extension thm. i.e. ($B(\mathcal{H})$ is injective)

Corollary 7.8. Let $T \in B(\mathcal{H})$. Let X be a spectral set for T.
Then TFAE:

1) T has a normal ∂X-dilation
2) X is a complete spectral set
3) $\tilde{\rho}$ is completely positive
4) $\tilde{\rho}$ has a positive extension.

Recall: spectral set (P17).

a Cmpt set X is called a spectral set of $T \in B(\mathcal{H})$ if $\sigma(T) \subseteq X$ and

$$r: R(X) \to B(\mathcal{H})$$
$$f(x) \mapsto f(T)$$

r is contractive.

Corollary 7.7 (Arveson). Let $\mathcal{B}$ be a C^*-alg.

$\mathcal{A} \subseteq \mathcal{B}$ be an opr. alg.

Let $\rho : \mathcal{A} \to B(\mathcal{H})$ be a unital homomorphism.

Let $\tilde{\rho} : \mathcal{A} + \mathcal{A}^* \to B(\mathcal{H})$ be the

$$a \mapsto \rho(a)$$
$$a^* \mapsto \rho(a)^*$$

Then TFAE:

1) ρ has a $\mathcal{B}$-dilation

2) ρ is completely contract

3) $\tilde{\rho}$ is completely positive.

If [illegible] has a $\mathcal{B}$-dilation

[illegible] $\to B(\mathcal{H})$

[illegible] $\varphi(b) = P_{\mathcal{H}} \pi(b)\big|_{\mathcal{H}}$

Normal ∂X-dilation (P47-48)

$T \in B(\mathcal{H})$ has a normal ∂X-dilation if there is a H.S. $K \supseteq \mathcal{H}$, and a normal opr $N \in B(K)$ s.t.

$$f(T) = P_{\mathcal{H}} f(N)\big|_{\mathcal{H}}$$

for f be a rational function on X

Ch 7: Arveson's extension thm. i.e. ($B(\mathcal{H})$ is injective)

Ch 7: Arveson's extension thm. i.e. ($\mathcal{B}(\mathcal{H})$ is injective)

Example. Parrott's example: every contractive $\not\Rightarrow$ completely contractive

$U, V \in \mathcal{B}(\mathcal{H})$. U is unitary. $UV \neq VU$

def $T_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

$T_2 = \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}$ $\in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$

$T_3 = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}$

Let $P(\mathbb{D}^3)$ be alg of polynomial with variables z_1, z_2, z_3

$\rho: z_i \mapsto T_i$, $i = 1, 2, 3$.

ρ is contractive but not completely contractive.

Corollary 7.7 (Arveson). Let $\mathcal{B}$ be a C^*-alg.

Normal ∂X-dilation (P47-48)

prop 7.9. (McAsey – Muhly).

Let $\mathcal{A}$ be the alg of upper triangular matrices.

$$\begin{pmatrix} 0 & & * \\ & \ddots & * \\ 0 & & \end{pmatrix}_{n\times n}$$

Let $\{E_{ij}\}$ be standard matrices for $M_n(\mathbb{C})$

if ρ is a rep of $\mathcal{A}$, with $\|\rho(E_{ij})\| \leq 1$, $i<j$, then ρ is completely contractive.