Completely bounded homomorphisms

Def: $T_1, T_2 \in B(H)$. T_1 is similar to T_2 if $T_1 = S^{-1} T_2 S$

Thm: A o.a, $\rho : A \to B(H)$, hom., S inv. s.t $\pi(\cdot) = S^{-1}\rho(\cdot)S$ c.c.
Then ρ is c.b. and $\|\rho\|_{cb} \le \|S^{-1}\| \cdot \|S\|$

Prf: $S_n = \begin{pmatrix} S & & 0 \\ & S & \\ & & \ddots \\ 0 & & & S \end{pmatrix}$, $(S_n)^{-1} = \begin{pmatrix} S^{-1} & & 0 \\ & \ddots & \\ 0 & & S^{-1} \end{pmatrix}$, $\|\rho_n\| \le \|S_n\| \cdot \|S_n^{-1}\| = \|S\| \, \|S^{-1}\|$

Thm: A o.a, $\rho: A \to B(H)$ a hom. ρ is c.b. and unital.
Then $\exists S \in B(H)$ with $\|S\| \cdot \|S^{-1}\| = \|\rho\|_{cb}$ s.t. $S^{-1}\rho(\cdot)S$ is c.c.

Pf: Let $A \subset B$, with B a C^*-alg. $\exists$ Hilbert space K, a $*$-hom $\pi: B \to B(K)$
and $V_1, V_2: H \to K$ with $\|V_1\| \cdot \|V_2\| = \|\rho\|_{cb}$ s.t. $\rho(a) = V_1^* \pi(a) V_2$

For $h \in H$

$$|h| = \inf\left\{ \left\| \sum_{i=1}^{n} \pi(a_i) V_2 h_i \right\| : \sum_{i=1}^{n} \rho(a_i) h_i = h,\ a_i \in A \text{ and } h_i \in H \right\}$$

$$\|V_1^*\|^{-1} \cdot \|h\| \leq |h| \leq \|V_2\| \cdot \|h\|$$

Completely bounded homomorphisms

Let $R : (H, |\cdot|) \to (H, \|\cdot\|)$ identity

$\|R^{-1}\| \cdot \|R\| \leq \|V_1^*\| \cdot \|V_2\|$ and $R^{-1} \rho(\cdot) R$

Let $a, a_1, \dots, a_n \in A$, $h, h_1, \dots, h_n \in H$ s.t. $h = \sum_{i=1}^{n} \rho(a_i) h_i$

Then $|\rho(a) h| = |\sum_{i=1}^{n} \rho(a a_i) h_i| \leq \|\sum_{i=1}^{n} \pi(a a_i) V_2 h_i\| \leq \|a\| \cdot \|\sum_{i=1}^{n} \pi(a_i) V_2 h_i\|$

So $|\rho(a) h| \leq \|a\| \cdot |h|$, so $\|\rho\| \leq 1$.

$n \in \mathbb{N}$, $\hat{A} = H^n$, $|\cdot|_n$ and $\|\cdot\|_n$ correspond to $|\cdot|$ and $\|\cdot\|_n$

$\rho : A \to B(H)$ a hom. ρ is c.b. and unital.

Then $\exists S \in B(H)$ with $\|S\|\cdot\|S^{-1}\| = \|\rho\|_{cb}$ s.t. $S^{-1}\rho(\cdot)S$ is c.c.

Pf: Let $A \subset B$, with B a C^*-alg. $\exists$ Hilbert space K, a $*$-hom $\pi: B \to B(K)$

Cor: A o.a. If $\rho: A \to B(H)$ is a unital c.b. hom., then

$$\|\rho\|_{cb} = \inf\{\|S\|\cdot\|S^{-1}\| : S^{-1}\rho(\cdot)S \text{ is c.c.}\}$$

Cor: Let A a unital C^*-alg and $\rho: A \to B(H)$ a unital bd. hom.

Then ρ is similar to a $*$-hom iff ρ is c.b.

This sim. S satisfies $\|S^{-1}\|\cdot\|S\| = \|\rho\|_{cb}$

Let G be l.c. top. grp. and let $L^\infty(G)$ denote the C^*-alg of bd. meas. functions.
For $g \in G$ let $R_g : L^\infty(G) \to L^\infty(G)$: $(R_g f)(g') = f(g'g)$.
A state m on $L^\infty(G)$ is called a right invariant mean (r.i.m) on G if
$$m(R_g f) = m(f)$$
Thm: G is l.c. grp, G amenable, and $\rho : G \to B(H)$ is s.c. hom, $\rho(e) = 1$ and $\|\rho\| = \sup\{\|\rho(g)\| : g \in G\} < \infty$. $\exists S \in B(H)$ with $\|S\| \cdot \|S^{-1}\| \le \|\rho\|^2$ s.t. $S^{-1}\rho(\cdot)S$ is a unitary representation.

Cor: A o.a. If $\rho : A \to B(H)$ is a unital c.b. hom., then

Let m an r.i.m.

$\langle x, y\rangle_1 = m(b_{x,y}) \quad \forall x, y \in H$: $b_{x,y}(g)$. given by $\langle \rho(g)x, \rho(g)y\rangle$ is bd. and cnt.

We will show: $\langle \cdot, \cdot\rangle_1 \sim \langle \cdot, \cdot\rangle$

Let $M = \|\rho\|$. Then $\forall g \in G$, $\rho(g)\rho(g)^* \leq M^2$ and $\rho(g)^*\rho(g) \leq M^2$

$\longrightarrow \rho(g^{-1})^*\rho(g^{-1}) \geq \frac{1}{M^2}$

$\longrightarrow \frac{1}{M^2}\langle x, x\rangle \leq b_{x,x}(g) \leq M^2\langle x, x\rangle$

$\longrightarrow \frac{1}{M^2}\langle x, x\rangle = m\left(\frac{1}{M^2}\langle x, x\rangle\right) \leq \underset{\langle x,x\rangle_1}{\underset{\parallel}{m(b_{x,x})}} \leq m(M^2\langle x, x\rangle) = M^2\langle x, x\rangle$

$\langle \rho(g)x, \rho(g)y \rangle_1 = m(R_g b_{x,y}) = m(b_{x,y}) = \langle x, y \rangle_1$

$R: (H, \langle \cdot, \cdot \rangle) \to (H, \langle \cdot, \cdot \rangle_1)$. Then $\|R\| \cdot \|R^{-1}\| \leq M^2$

Thm: G is l.c. grp, G amenable, and $\rho: G \to B(H)$ is s.c. hom, $\rho(e) = 1$ and $\|\rho\| = \sup\{\|\rho(g)\| : g \in G\} < \infty$. $\exists\, S \in B(H)$ with $\|S\| \cdot \|S^{-1}\| \leq \|\rho\|^2$ s.t. $S^{-1}\rho(\cdot)S$ is a unitary representation.

$\exists$ m an i.l.m. $\forall x, y \in H$: $b_{x,y}(g)$, given by $\langle \rho(g)x, \rho(g)y \rangle$ is bd. and cnt.

Thm: A o.a., $\rho: A \to B(H)$ hom.
ρ c.b. $\iff \exists S \in B(H)$ s.t.
$\|S\| \cdot \|S^{-1}\| = \|\rho\|_{cb}$ and
$S^{-1}\rho(\cdot)S$ is c.c.

Cor: A C^*-alg, $\rho: A \to B(H)$ hom
bd. unital hom. ρ is similar
to $*$-hom iff ρ is c.b.
Then the sim. S satisfies
$\|S^{-1}\| \cdot \|S\| = \|\rho\|_{cb}$

Thm: G amenable grp,
$\rho: G \to B(H)$ s.c. hom.
with $\rho(e) = 1$ s.t.
$\|\rho\| = \sup\{\|\rho(g)\| : g \in G\}$
is finite. $\exists S \in B(H)$
with $\|S^{-1}\| \cdot \|S\| \leq \|\rho\|^2$
s.t. $S^{-1}\rho(\cdot)S$ is a unitary
representation.

Kadison's conjecture

Let A a C^*-alg, $\rho: A \to B(H)$ a
bd. unital hom.
Then ρ is similar to a $*$-hom

Suppose that all unif. bd. s.c.
representations of a l.c. grp G are
unitarisable.
Is G then necessarily amenable?

Cor. Let $T \in B(H)$ be invertible s.t. $\|T^k\| \le M \quad \forall k \in \mathbb{Z}$.
Then $\exists\, S \in B(H)$ with $\|S\| \cdot \|S^{-1}\| \le M^2$ s.t. $S^{-1} T S$ is unitary.

Pf: $\rho : \mathbb{Z} \to B(H)$

$\rho(k) = T^k$.

Then $\exists\, S \in B(H)$ s.t. $S^{-1} \rho(\cdot) S$ is a unitary rep.
$\to S^{-1} T^{1} S$ is unitary

Lemma Let A, B C^*-alg,
$\rho : A \to B$ a unital hom.
If ρ maps unitaries to unitaries, then ρ is a $*$-hom.

Pf: $\rho(u^*) = \rho(u^{-1}) = \rho(u)^{-1} = \rho(u)^*$ $\forall u$ unitary. $\forall h \in A$, $h = h^*$ and $\|h\| \le 1$ we have $h \pm i\sqrt{1-h^2}$ is unitary.

So ρ is a $*$-hom.

Thm: A o.a., $\rho : A \to B(H)$ hom. ρ c.b. $\Leftrightarrow$ $\exists S \in B(H)$ s.t. $\|S\| \cdot \|S^{-1}\| = \|\rho\|_{cb}$ and $S^{-1}\rho(\cdot)S$ is c.c.

Cor: A C^*-alg, $\rho : A \to B(H)$ bd. unital hom. ρ is similar to $*$-hom iff ρ is c.b. Then the sim. S satisfies $\|S^{-1}\| \cdot \|S\| = \|\rho\|_{cb}$

Thm: G amenable grp, $\rho : G \to B(H)$ s.c. hom. with $\rho(e) = 1$ s.t. $\|\rho\| = \sup\{\|\rho(g)\| : g \in G\}$ is finite. $\exists S \in B(H)$ with $\|S^{-1}\| \cdot \|S\| \le \|\rho\|^2$ s.t. $S^{-1}\rho(\cdot)S$ is a unitary representation.

Let A a comm. C^*-alg. If $\rho : A \to B(H)$ is a bd hom, then ρ is c.b. and $\|\rho\|_{cb} \le \|\rho\|^2$.

Pf: Let $G = \{u \in A \mid u \text{ unitary}\}$. $\exists S \in B(H)$ s.t. $\|S\| \cdot \|S^{-1}\| \le \|\rho\|^2$ and $S^{-1}\rho(u)S$ is unitary for all $u \in G$. So $S^{-1}\rho(\cdot)S$ is a $*$-hom. So ρ is c.b. and $\|\rho\|_{cb} \le \|\rho\|^2$

Cor. Let $T \in B(H)$ be invertible

Then $\exists S \in B(H)$ s.t.

$S^{-1}\rho(\cdot)S$ ...

Lemma Let A, B C^*-alg, $\rho : A \to B$ a unital hom

Cor. Let $T \in B(H)$. T is similar to a self-adjoint operator iff $\exists$ interval $[a,b]$ and constant K s.t. for all polynomials p we have

$$\|p(T)\| \le K \cdot \sup\{|p(t)| : t \in [a,b]\}$$

Pf $\Rightarrow$ $T = S^{-1} A S$

$$\|p(T)\| = \|S^{-1} p(A) S\|$$

($\Leftarrow$) Let $B = C([a,b])$

$\phi : B \to B(H)$:

$\phi(p) = p(T)$ for polynomials p.

So $\exists S \in B(H)$ s.t. $S^{-1}\phi(\cdot)S$ is a $*$-hom.

$S^{-1} T S$ is self-adjoint.

Let A a C^*-alg and $\pi : A \to B(H)$ a unital $*$-hom. A linear map $\delta : A \to B(H)$ is called a derivation if

$$\delta(ab) = \pi(a)\delta(b) + \delta(a)\pi(b)$$

δ is called inner if $\delta(a) = \pi(a)X - X\pi(a)$ for some $X \in B(H)$.

So δ is inner.

Thm: A o.a, $\rho: A \to B(H)$ hom.
ρ c.b. $\iff \exists S \in B(H)$ s.t.
$\|S\| \cdot \|S^{-1}\| = \|\rho\|_{cb}$ and
$S^{-1}\rho(\cdot)S$ is c.c.

Cor: A C^*-alg, $\rho: A \to B(H)$ bd. unital hom. ρ is similar to $*$-hom iff ρ is c.b.
Then the sim. S satisfies
$\|S^{-1}\| \cdot \|S\| = \|\rho\|_{cb}$

Thm: G amenable grp, $\rho: G \to B(H)$ s.c. hom. with $\rho(e) = 1$ s.t.
$\|\rho\| = \sup\{\|\rho(g)\| : g \in G\}$
is finite. $\exists S \in B(H)$
with $\|S^{-1}\| \cdot \|S\| \leq \|\rho\|^2$
s.t. $S^{-1}\rho(\cdot)S$ is a unitary representation.

Given δ, define $\rho: A \to B(H \oplus H)$ by
$$\rho(a) = \begin{pmatrix} \pi(a) & \delta(a) \\ 0 & \pi(a) \end{pmatrix}.$$
ρ is a hom.

Prop: A derivation δ is inner iff ρ is similar to a $*$-hom.

Pf: $(\Rightarrow)$ $\delta(a) = \pi(a)X - X\pi(a)$.
Set $S = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$, $S^{-1} = \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix}$

$$S\rho(a)S^{-1} = \begin{pmatrix}\pi(a) & \pi(a)X\\ 0 & \pi(a)\end{pmatrix}\begin{pmatrix}1 & -X\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\pi(a) & 0\\ 0 & \pi(a)\end{pmatrix}.$$

($\Leftarrow$) Let S sim. s.t. $\gamma(\cdot) = S^{-1}\rho(\cdot)S$ is a $*$-hom.

Set $X = SS^*$. $\rho(a)X = S\gamma(a)S^* = (S\gamma(a^*)S^*)^*$

$= (\rho(a^*)SS^*)^* = X\rho(a^*)^*$

Write $X = \begin{pmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{pmatrix} \to \delta(a) = X_{12}X_{22}^{-1}\pi(a) - \pi(a)X_{12}X_{22}^{-1}$,

so δ is inner.

Cor: A derivation δ is inner iff δ is completely bounded.

Thm: Let $T \in B(H)$. T is similar to a contraction iff the hom. $\rho: P(\mathbb{D}) \to B(H)$ given by $\rho(p) = p(T)$ is c.b.

to $*$-hom iff ρ is c.b. | s.t. $S^{-1}\rho(\cdot)S$ is a unitary, | Set $S = \begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix}$, $S^{-1} = \begin{pmatrix}1 & -X\\ 0 & 1\end{pmatrix}$

$S\rho(a)S^{-1} = \begin{pmatrix} \pi(a) & \pi(a)X \\ 0 & \pi(a) \end{pmatrix} \cdot \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi(a) & 0 \\ 0 & \pi(a) \end{pmatrix}.$

$(\Leftarrow)$ Let S inv. s.t. $\gamma(\cdot) = S^{-1}\rho(\cdot)S$ is a $*$-hom.

Set $X = SS^*$. $\rho(a)X = S\gamma(a)S^* = (S\gamma(a^*)S^*)^*$

$= (\rho(a^*)SS^*)^* = X\rho(a^*)^*$.

Write $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \rightarrow \delta(a) = X_{12}X_{22}^{-1}\pi(a) - \pi(a)X_{12}X_{22}^{-1}$,

so δ is inner.

Cor: A derivation δ is inner iff δ is completely bounded.

Thm: Let $T \in B(H)$.
T is similar to a contraction iff the hom. $\rho : P(\mathbb{D}) \to B(H)$ given by $\rho(p) = p(T)$ is c.b.

Pf: $(\Leftarrow)$ ρ c.b., then $\exists S \in B(H)$ s.t. $\|S\| \cdot \|S^{-1}\| = \|\rho\|_{cb}$ s.t. $S^{-1}\rho(\cdot)S$ is c.c.

So $\|S^{-1}TS\| = \|S^{-1}\rho(z)S\| \leq \|z\| = 1$

$\Longleftrightarrow \exists S \in B(H)$ s.t.
$\|S^{-1}\| = \|\rho\|_{cb}$ and
$\rho(\cdot)S$ is c.c.

C^*-alg, $\rho: A \to B(H)$
al hom. ρ is similar
om iff ρ is c.b.
m. S satisfies
$\| = \|\rho\|_{cb}$

Thm: G amenable grp, $\rho: G \to B(H)$ s.c. hom. with $\rho(e) = 1$ s.t. $\|\rho\| = \sup\{\|\rho(g)\| : g \in G\}$ is finite. $\exists S \in B(H)$ with $\|S^{-1}\| \cdot \|S\| \leq \|\rho\|^2$ s.t. $S^{-1}\rho(\cdot)S$ is a unitary representation.

($\Rightarrow$) $R = S^{-1}TS$ is a contraction
Let $\theta: p \to p(R)$. $\exists K, P_H, U$ s.t. $R^n = P_H U^n$. $\varphi: C(\bar{\mathbb{D}}) \to B(K)$ by $\varphi(f) = f(U)$. Then

$$\theta_n(P_{ij}) = \begin{pmatrix} P_H & & 0 \\ & \ddots & \\ 0 & & P_H \end{pmatrix} \varphi_n(P_{ij}).$$

φ is positive, $C(\bar{\mathbb{D}})$ is comm $\Rightarrow \varphi$ c.p.
$\Rightarrow \|\varphi\|_{cb} = \|\varphi\| = \|\varphi(1)\| = 1$. $\rho(\cdot) = S\theta(\cdot)S^{-1}$