Let H be Hilbert space (over $\mathbb{C}$), $T \in B(H)$

Def. • T is similar to a contraction: $\exists\, S, C \in B(H)$
S invertible, C contraction, s.t. $T = S^{-1} C S$.

• T is power bounded: $\sup_{n \geq 0} \|T^n\| < \infty$.

• T is polynomially bounded: $\exists\, K$ s.t.
$\forall p \in \mathcal{P}$: $\|p(T)\| \leq K \|p\|_\infty$, where
$\mathcal{P} = \{ z \longmapsto c_0 + c_1 z + \dots + c_n z^n : \mathbb{D} \to \mathbb{C},\ c_1, \dots, c_n \in \mathbb{C} \}$,
$\|p\|_\infty = \sup_{z \in \mathbb{D}} |p(z)|$

$\mathbb{D}$ ← open unit disk in $\mathbb{C}$.

Note polyn. bdd $\Rightarrow$ power bdd.

If $T = S^{-1} C S$, then $T^n = S^{-1} C^n S$,
$\|T^n\| \leq \|S^{-1}\| \, \|C^n\| \, \|S\| = \|S^{-1}\| \, \|S\|$

So T is power bdd

Question (Béla Nagy): T power bdd $\Rightarrow$ T similar to contraction?

Foguel (1964): No.

Von Neumann inequality: $\|p(C)\| \leq \|p\|_\infty$

So $\|p(T)\| = \|S^{-1} p(C) S\| \leq \|S^{-1}\| \, \|S\| \, \|p\|_\infty$.

So T is polyn. bdd.

Conjecture (Halmos, 1970): T polyn bdd $\Rightarrow$ T similar to a contraction.

Pisier (1997): No.

Recall **Thm 9.11**: T similar to contraction $\Leftrightarrow$ $p \mapsto p(T)$ is completely bounded.

Th. 10.1 Let $T \in B(H)$ be compact. Then:

T similar to contraction $\Leftrightarrow$ T is power bdd.

Pf $\Rightarrow$) Clear.

$\Leftarrow$) $M = \sup_n \|T^n\|$. Then spectral radius

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \leq \lim_{n\to\infty} M^{1/n} = 1$$

so $\sigma(T) \subseteq \overline{\mathbb{D}}$ and as T is compact

$\sigma(T) \cap \underbrace{\mathbb{T}}_{\{z \in \mathbb{C} : |z| = 1\}}$ is finite.

$\sigma(T) \cap \mathbb{T}$ and $\sigma(T) \setminus \mathbb{T}$ are clopen in $\sigma(T)$.

Then $\exists$ idempotent E st. $ET = TE$ and

$$\sigma(ET|_{E(H)}) = \sigma(T) \cap \mathbb{T}, \quad \sigma((I-E)T|_{(I-E)(H)}) = \sigma(T) \setminus \mathbb{T}$$

Conjecture (Halmos, 197?): T polyn bdd $\Rightarrow$ T similar to a

$\rightsquigarrow \quad S^{-1}TS = T_1 \oplus T_2, \quad T_1 : E(H) \to E(H)$

$\uparrow$ finite dim.

T is power bdd $\Rightarrow$ $S^{-1}TS$ power bdd $\Rightarrow T_1$ power bdd.

Let $J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ be a Jordan block of T_1.

Then $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & & \\ & \ddots & \ddots & \\ & & \ddots & n\lambda^{n-1} \\ & & & \lambda^n \end{pmatrix}$, $|\lambda| = 1$. So must $|\lambda| < 1$.

So T_1 is similar to a contraction.

Th. 10.1 Let $T \in B(H)$ be compact. Then:

We have $\sigma(T_2) \subseteq \mathbb{D}$

By Rota' thm (Cor 9.14): T_2 is similar to contraction. Hence T similar to contraction. $\square$

Foguel-Hankel operators

$\ell^2(H) = \{ (h_0, h_1, h_2, \dots) : h_k \in H : \sum_{k=0}^{\infty} \|h_k\|^2 < \infty \}$

$S := S_H : \ell^2(H) \to \ell^2(H)$,

$S(h_0, h_1, \dots) = (0, h_0, h_1, \dots)$ right shift

and

$S^*(h_0, h_1, \dots) = (h_1, h_2, \dots)$ left shift.

Def For $X = (A_{ij})_{i,j \geq 0}$ with $A_{ij} \in B(H)$, the operator $F = \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix}$ on $\ell^2(H) \oplus \ell^2(H)$ is called a <u>Foguel operator</u> over H with <u>symbol</u> X.

If $\forall i,j$ $A_{ij} = A_{i+j} := A_{i+j,0}$ for all

then $X = (A_{i+j})_{i,j}$ is <u>Hankel operator</u> and

F is a Foguel-Hankel operator.

Rem. $X = (A_{i+j})_{i,j}$ Hankel $\Longrightarrow$

$$S^*X = \begin{pmatrix} 0 & I & 0 & \cdots \\ 0 & 0 & I & 0 \cdots \\ & & & \ddots \end{pmatrix}\begin{pmatrix} A_0 & A_1 & A_2 & \cdots \\ A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & \cdots & \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & \cdots \\ A_2 & A_3 & \cdots \\ \vdots & & \end{pmatrix}$$

$$XS = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} 0 & & \\ I & 0 & \\ 0 & I & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & \\ A_2 & A_3 & \\ & & \ddots \end{pmatrix}$$

So $S^*X = XS = (A_{i+j+1})_{i,j \geq 0}$.

So $S^{*j}X = XS^j$

Note $F^2 = \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix}\begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix} = \begin{pmatrix} S^{*2} & S^*X + XS \\ 0 & S^2 \end{pmatrix}$

Def For $X = (A_{ij})_{i,j \geq 0}$ with $A_{ij} \in B(H)$, the operator $F = \begin{pmatrix} S^* & X \\ \ldots \end{pmatrix}$ on $\ell^2(H) \oplus \ell^2(H)$ is called

$$F^n = \begin{pmatrix} S^{*n} & X_n \\ 0 & S^n \end{pmatrix}, \quad X_n = \sum_{j=0}^{n-1} S^{*j} X S^{n-1-j}, \quad X_0 := 0, \; X_1 = X.$$

Hence $X_n = \sum_{j=0}^{n-1} X S^j S^{n-1-j} = n X S^{n-1} = (A_{i+j+n-1})_{i,j \geq 0}$.

Rem $\|S^{*n}\| = \|S^n\| = 1$

So F power bdd $\iff \sup_{n \geq 0} \|X_n\| < \infty$.

- $\delta : \mathcal{P} \longrightarrow B(\ell^2(H))$

$\delta(c_0 + c_1 z + \ldots + c_n z^n) = c_0 X_0 + c_1 X_1 + \ldots + c_n X_n$

δ is linear.

Rem: $X = (A_{i+j})_{i,j}$ Hankel $\Rightarrow$ $(A \; A \; A_2 \ldots) = (A_1 \; A_2 \ldots)$

Rem $X = (A_{i+j})_{i,j \geq 0}$ Hankel

$\Rightarrow X_n = n(A_{i+j+n-1})_{i,j \geq 0}$

$\Rightarrow \delta(p)_{ij} = c_1 A_{i+j} + c_2 2 A_{i+j+1} + c_3 3 A_{i+j+2} + \ldots + c_n n A_{i+j+n-1}$

Rem $F^n = \begin{pmatrix} S^{*n} & X_n \\ 0 & S^n \end{pmatrix}$, $= (X p'(S))_{i,j \geq 0}$ is Hankel

so $p(F) = \begin{pmatrix} p(S^*) & \delta(p) \\ 0 & p(S) \end{pmatrix}$

By von Neumann ineq: (S is contraction)

$\|p(S)\| \leq \|p\|_\infty$, $\|p(S^*)\| \leq \|p\|_\infty$.

So F polyn bdd $\iff \exists M : \|\delta(p)\| \leq M \|p\|_\infty$

i.e. δ is <u>bdd</u> linear

Rem. F completely polyn. bdd $\iff$ $\delta : \mathcal{D} \to \mathcal{B}(\ell^2(H))$ completely bdd.

Pf — " — $\overset{\text{def}}{\iff}$ $\rho : \mathcal{D} \to \mathcal{B}(H)$ given by $\rho(p) = p(F)$ is completely bdd i.e.

$$\forall n \quad \rho_n \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} = \begin{pmatrix} \rho(p_{11}) & \cdots & \rho(p_{1n}) \\ \vdots & & \\ \rho(p_{n1}) & & \rho(p_{nn}) \end{pmatrix}$$

$\iff \forall n$ $\quad \rho_n : M_n(\mathcal{D}) \to M_n(\mathcal{B}(H))$ is bdd

$$\begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \\ p_{n1} & & p_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} p_{11}(S^*) & \delta(p_{11}) & & \\ 0 & p_{11}(S) & \cdots & - \\ & \vdots & & \\ & \vdots & & \end{pmatrix} \text{ is bdd}$$

$\overset{\text{Canonical}}{\iff}$ shift (Chap 8 p 57)

$$\begin{pmatrix} p_{11} & & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & & p_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} p_{11}(S^*) & \cdots & p_{1n}(S^*) & \delta(p_{11}) & \cdots & \delta(p_{1n}) \\ \vdots & & & \vdots & & \\ p_{n1}(S^*) & \cdots & p_{nn}(S^*) & \delta(p_{n1}) & & \delta(p_{nn}) \\ & & & & & \\ & 0 & & p_{11}(S) & & - \\ & & & \vdots & & \\ & & & \vdots & & \end{pmatrix}$$

is bdd

S^*, S are (similar to contr.) so by Th 9.11

$p \longmapsto p(S^*)$, $p \longmapsto p(S)$ are completely bdd

So

$$\iff \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} \delta(p_{11}) & \cdots & \delta(p_{1n}) \\ \vdots & & \\ \delta(p_{n1}) & \cdots & \delta(p_{nn}) \end{pmatrix} \text{ bdd} \iff \delta \text{ completely bdd.}$$

Rem. $\delta(pq) = p(S^*)\delta(q) + \delta(p)q(S)$.

Pf $(pq)(F) = \begin{pmatrix} pq(S^*) & \delta(pq) \\ 0 & pq(S) \end{pmatrix}$

also $= p(F)q(F) = \begin{pmatrix} p(S^*) & \delta(p) \\ 0 & p(S) \end{pmatrix}\begin{pmatrix} q(S^*) & \delta(q) \\ 0 & q(S) \end{pmatrix}$

$= \begin{pmatrix} * & p(S^*)\delta(q) + \delta(\mathrm{p})q(S) \\ 0 & * \end{pmatrix}$ $\square$

Prop. Let H, K Hilbert spaces, $\Phi : B(H) \to B(K)$ bdd lin.

$A_i \in B(H)$, $i \geq 0$ st $A = (A_{i+j})_{i,j\geq 0}$ is bdd lin on $\ell^2(H)$

Then $A_\Phi = (\Phi(A_{i+j}))_{i,j\geq 0}$ is bdd lin Hankel operator and $\|A_\Phi\| \leq \|\Phi\|\,\|A\|$.

Pf By Nehari-Page Thm (Th 5.10): $\exists$ seq $(A_n)_{n=-\infty}^{-1}$ st.

$$\|A\| = \sup_{r<1} \Big\| \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} A_n \Big\|_\infty$$

sup over $\theta \in \mathbb{T}$.

By Nehari–Page Thm again,

$$\|A_\Phi\| = \inf_{(B_n)_{n=-\infty}^{-1} \text{ in } B(K)} \sup_{r<1} \Big\| \sum_{n=-\infty}^{-1} r^{|n|} e^{in\theta} B_n + \sum_{n=0}^{\infty} r^{|n|} e^{in\theta} \Phi(A_n) \Big\|_\infty$$

Take $B_n = 0$

$$\leq \Big\| \Phi\Big(\sum_{n=0}^{\infty} \cdots \Big) \Big\| \leq \|\Phi\| \, \|A\|.$$

$\square$

<u>Prop 10.4</u>. $\Phi : B(H) \to B(K)$ bdd lin.

$X = (A_{i+j})_{i,j\geq 0}$ bdd lin on $\ell^2(H)$

$$F = \begin{pmatrix} S_H^* & X \\ 0 & S_H \end{pmatrix}, \quad F_\Phi = \begin{pmatrix} S_K^* & \overset{= (\Phi(A_{i+j}))_{i,j\geq 0}}{\Phi(X)} \\ 0 & S_K \end{pmatrix}$$

Then (i) F is power bdd $\Rightarrow$ F_Φ power bdd

(ii) F polyn bdd $\Rightarrow$ F_Φ polyn bdd.

<u>Prf</u> of (ii): By earlier Remarks:

F polyn bdd $\Longleftrightarrow$ $\delta : \mathcal{P} \to B(\ell^2(H))$ is bdd

$\delta(p) = X p'(S_H)$

$\delta_\Phi(p) := X_\Phi p'(S_K)$.

Then

$$\delta(p) = \overbrace{X p'(S_H)}^{\text{is Hankel}} = (B_{i+j})_{i,j\geq 0}$$

then $\delta_\Phi(p) = \big(\Phi(B_{i+j})_{i,j\geq 0} \big)$.

So $\|\delta_\Phi(p)\| \overset{\text{Prop 10.3}}{\le} \|\Phi\| \, \|(B_{i+j})_{i,j\ge 0}\|$

$= \|\Phi\| \, \|\delta(p)\| \le \|\Phi\| M \|p\|_\infty$

So δ_Φ is bdd so F_Φ is polyn. bdd. □

Pisier's example

$V := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $C = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$, $I_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

For $n \ge 1$ define

$1 \le i \le n-1$: $C_{i,n} = V^{\otimes i} \otimes C \otimes I_2^{\otimes(n-i-1)} \in \underbrace{M_2 \otimes \cdots \otimes M_2}_{n} \cong M_{2^n}$

$i \ge n$: $C_{i,n} = 0$.

Then $C_{i,n}^* C_{i,n} + C_{i,n} C_{i,n}^* = I$ for $i < n$

and

$$\frac{1}{2}\sum_{i=0}^{n} |a_i| \le \Big\| \sum_{i=0}^{n-1} a_i C_{i,n} \otimes C_{i,n} \Big\| \le \sum_{i=0}^{n-1} |a_i|$$

for all $a_0, \ldots, a_{n-1} \in \mathbb{C}$.

Consider $\bigoplus_{n=1}^{\infty} \mathbb{C}^{2^n}$ Hilbert direct sum and define

$W_i = \bigoplus_{n=1}^{\infty} C_{i,n}$, i.e. $W_i \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} C_{i,1} v_1 \\ C_{i,2} v_2 \\ \vdots \end{pmatrix}$

eventually 0

$\|A_\Phi\| = \inf \ldots \sup_{r<1} \| \sum_{n=-\infty} r^{|n|} e^{in\theta} B_n \|$

Then (i) F n power bdd $\Rightarrow$ F_Φ power bdd

(ii) F polyn bdd $\Rightarrow$ F_Φ polyn bdd

Then W_i bdd lin and

(a) $\|\sum_{i=0}^{\infty} a_i W_i\|^2 = \sum_{i=0}^{\infty} |a_i|^2$

(b) $\frac{1}{2}\sum_{i=0}^{n-1} |a_i| \le \|\sum_{i=0}^{n-1} a_i\, c_{i,n} \otimes W_i\| \le \sum_{i=0}^{n-1} |a_i|$

(c) $W_i W_j + W_j W_i = 0$

(d) $W_i W_j^* + W_j^* W_i = \delta_{ij}(I - P_i)$ where P_i is some orth projection in $\bigoplus \mathbb{C}^{2^n}$ of finite rank.

Th 10.5 (Pisier) Let $(a_n)_{n=0}^{\infty}$ in $\mathbb{C}$,

$H = \bigoplus_{n=1}^{\infty} \mathbb{C}^{2^n}$, $X := (a_{i+j} W_{i+j})_{i,j\ge 0}$ on $\ell^2(H)$

and $F = \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix}$.

TFAE:

(i) F is polyn bdd

(ii) F is power bdd

(iii) $\sup_n n \sum_{k=n-1}^{\infty} |a_k|^2 < \infty$

If $\sum_{k=0}^{\infty} (k+1)^2 |a_k|^2 = \infty$ then F is not similar to a contraction.

Then W_i bdd lin and

(a) $\|\sum^{\infty} a_i W_i\|^2 = \sum^{\infty} |a_i|^2$

Pf. (i) $\Rightarrow$ (ii) clear.

(ii) $\Rightarrow$ (iii) $X_n := n X S^{n-1}$ Then

F power bdd $\iff \sup_n \|X_n\| < \infty$.

$X_n = n X S^{n-1} = \left(n\, a_{i+j+n-1} W_{i+j+n-1}\right)_{i,j\geq 0}$

So

$X_n^* X_n + X_n X_n^* = \left(n^2 \sum_{h=0}^{\infty} \overline{a}_{i+h+n-1} a_{j+h+n-1} \left(W^*_{i+h+n-1} W_{j+h+n-1} + W_{i+h+n-1} W^*_{j+h+n-1}\right)\right)_{i,j}$

By (c), this is diagonal matrix.

So $\|X_n^* X_n + X_n X_n^*\| =$ norm of entry$_{00}$ =

$= n^2 \|\sum_{h=0}^{\infty} |a_{h+n-1}|^2 (V^*_{h+n-1} V_{h+n-1} + W_{h+n-1} V^*_{h+n-1})\|$

$= \sup_m n^2 |\sum_{h=0}^{\infty} |a_{h+n-1}|^2 \underbrace{(C^*_{h+n-1,m} C_{h+n-1,m} + C_{h+n-1,m} C^*_{h+n-1,m})}_{I \quad h+n-1<m}|$

$= n^2 \sum_{h=0}^{\infty} |a_{h+n-1}|^2$

Hence

$\sup_n n^2 \sum_{h=0}^{\infty} |a_{h+n-1}|^2 = \sup_n \|X_n^* X_n + X_n X_n^*\|$

$\leq \sup_n 2\|X_n\|^2 < \infty.$

Th 10.5 (Pisier) Let $(a_n)_{n=0}^{\infty}$ in $\mathbb{C}$,

(iii) $\Rightarrow$ (i) Define $\Phi : B(\ell^2) \longrightarrow B(H)$

$$\Phi\left((a_{i,j})_{i,j \geq 0} \right) = \sum_{i=0}^{\infty} a_{i,0} W_i .$$

Then

$$\| \Phi ((a_{i,j})_{i,j\geq 0}) \|^2 = \| \sum_{i=0}^{\infty} a_{i,0} W_i \|^2 \overset{(a)}{=} \sum_{i=0}^{\infty} |a_{i,0}|^2$$

$$\leq \| (a_{ij})_{i,j} \|^2 .$$

So $\|\Phi\| \leq 1$.

As $\sup_n n \sum_{k=0}^{\infty} |a_{k+n-1}|^2 < \infty$, Th 10.2 yields that

$(a_{i+j} E_{i+j})_{i,j \geq 0}$ is polyn bdd. So by Prop 10.4

$(a_{i+j} \Phi(E_{i+j}))_{i,j \geq}$ is polyn bdd. and $\Phi(E_{i+j}) = W_{i+j}$

Th 10.5 (Pisier) Let $(a_n)_{n=0}^{\infty}$ in $\mathbb{C}$.

$H = \bigoplus_{n=1}^{\infty} \mathbb{C}^{2^n}$, $X := (a_{i+j} W_{i+j})_{i,j \geq 0}$

and $F = \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix}$ on $\ell^2(H)$.

TFAE:

(i) F is polyn bdd

(ii) F is power bdd

(iii) $\sup_n n \sum_{k=n-1}^{\infty} |a_k|^2 < \infty$

If $\sum_{k=0}^{\infty} (k+1)^2 |a_k|^2 = \infty$ then F is not similar to a contraction.

Assume $\sum (k+1)^2 |a_k|^2 = \infty$.

We show $\delta : \mathcal{D} \to B(\ell^2(H))$ not completely bdd

then F not compl. polyn. bdd

Then by Thm 9.11 F is not similar to a contr.

$$\delta(z^n) = X_n = \left(n\, a_{i+j+n-1} W_{i+j+n-1}\right)_{i,j \geq 0}.$$

So $(\delta(z^n))_{00} = n\, a_{n-1} W_{n-1}$

Suffices: $\delta_0 : \mathcal{D} \to B(H)$ $\delta_0(c_0 + c_1 z + \dots + c_n z^n)$

$= c_1 a_0 W_0 + \dots + c_n a_{n-1} W_{n-1}$ $\forall i,j$

is not compl. bdd.

Take $p(z) = \sum_{i=0}^{n} (i+1)\, \bar{a}_i\, C_{i,n}\, z^{i+1}$ $2^n \times 2^n$ matrix valued polyn.

Then

$$\delta_0^{(2^n)}(p) = \sum_{i=0}^{n} (i+1)^2 |a_i|^2\, C_{i,n} \otimes \underbrace{W_i}_{C_{i,n}}$$

So

$$\|\delta_0^{(2^n)}(p)\| \geq \| \text{——} \cdots \text{——} \|$$

$$\geq \frac{1}{2} \sum_{i=0}^{n} (i+1)^2 |a_i|^2$$

$$= \frac{1}{2} \left(\sum_{i=0}^{\infty} (i+1)^2 |a_i|^2 \right)^{1/2} \|p\|_\infty$$

So $\|\delta_0\|_{cb} \geq \frac{1}{2} \left(\sum_{i=0}^{n} (i+1)^2 |a_i|^2 \right)^{1/2} \nearrow \infty$ as $n \to \infty$

$\square$

Then

$$\|\Phi((a_{ij})_{i,j\geq 0})\|^2 = \left\| \sum_{i=0} a_{i,0} v_i \right\|^2 = \sum_{i=0} |a_{i,0}|^2$$

and $F = (S^* \quad X)$. on $\ell^2(H)$

Now take $a_i = \begin{cases} 2^{-k} & \text{if } i = 2^k - 1 \\ 0 & \text{otherwise} \end{cases}$

or $a_i = \dfrac{1}{(i+1)^{3/2}}$.

Then F is polyn bdd but not similar to a contraction!

---

so $\|\Phi\| \leq 1$. $\leq \|(a_{ij})_{i,j}\|^2$

As $\sup_n n \sum_{k=0}^{\infty} |a_{k+n-1}|^2 < \infty$, Th 10.2 yields that

Th 10.5 (Pisier) Let $(a_n)_{n=0}^{\infty}$ in $\mathbb{C}$,

$H = \bigoplus_{n=1}^{\infty} \mathbb{C}^{2^n}$, $X := (a_{i+j} W_{i+j})_{i,j \geq 0}$

and $F = \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix}$. on $\ell^2(H)$

TFAE:

(i) F is polyn bdd

(ii) F is power bdd

(iii) $\sup_n n \sum_{k=n-1}^{\infty} |a_k|^2 < \infty$