$\mathcal{A}, \mathcal{B}$ unital C^*-algs.

$\leadsto \mathcal{A} \otimes_{alg} \mathcal{B}$

Def.: A norm $\|\cdot\|_\gamma$ on $\mathcal{A} \otimes_{alg} \mathcal{B}$ is a cross-norm if

1) $\|a \otimes b\|_\gamma = \|a\| \|b\|$

it is a C^*-cross-norm if

2) $\|xy\|_\gamma \leq \|x\|_\gamma \|y\|_\gamma$

3) $\|x^* x\|_\gamma = \|x\|^2 = \|x^*\|^2$,

$\forall x, y \in \mathcal{A} \otimes \mathcal{B}$

Spatial tensor prod. for Hilb. sp. ops.

Recall: $\mathcal{H}, \mathcal{K}$ Hilbert sp.

Equip $\mathcal{H} \otimes_{alg} \mathcal{K}$ with the scalar prod.:

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle := \langle \xi, \xi' \rangle_{\mathcal{H}} \langle \eta, \eta' \rangle_{\mathcal{K}}$$

$\mathcal{H} \otimes \mathcal{K}$ = compl. of $\mathcal{H} \otimes_{alg} \mathcal{K}$ w.r.t. $\langle \cdot, \cdot \rangle$

Let $T \in \mathcal{B}(\mathcal{H})$, $S \in \mathcal{B}(\mathcal{K})$

$T \otimes_{sp} S (\xi \otimes \eta) := T\xi \otimes S\eta$

$\leadsto T \otimes_{sp} S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$

$\leadsto (T \otimes_{sp} S)(T' \otimes_{sp} S') = TT' \otimes_{sp} SS'$

Def.:

$\mathcal{A}, \otimes_{s\ldots}$

$(T \otimes_{sp} S)^* = T^* \otimes_{sp} S^*$

Def.: $\mathcal{A}_i \subseteq B(\mathcal{H}_i)$, $i = 1,2$

$\mathcal{A}_1 \otimes_{sp} \mathcal{A}_2 := \overline{\text{span}} \{ T \otimes_{sp} S \} \subseteq B(\mathcal{H} \otimes \mathcal{K})$

Note:

$$\begin{array}{ccc} \mathcal{A}_1 \otimes_{alg} \mathcal{A}_2 & \xrightarrow{1-1} & \mathcal{A}_1 \otimes_{sp} \mathcal{A}_2 \\ \cap\!| & & \cap\!| \\ B(\mathcal{H}_1) \otimes B(\mathcal{H}_2) & & B(\mathcal{H}_1 \otimes \mathcal{H}_2) \end{array}$$

Def.:

$$\left\| \sum T_i \otimes_{sp} S_i \right\|_{sp} := \left\| \sum T_i \otimes_{sp} S_i \right\|$$

$\rightsquigarrow$ C^*-cross-norm!

Def.: $\pi_i : \mathcal{A}_i \longrightarrow B(\mathcal{H}_i)$, $i = 1,2$ unital $*$-hom

$\pi_1 \otimes \pi_2 : \mathcal{A}_1 \otimes_{alg} \mathcal{A}_2 \longrightarrow B(\mathcal{H}_1 \otimes \mathcal{H}_2)$

$\pi_1 \otimes \pi_2 (a_1 \otimes a_2) := \pi_1(a_1) \otimes_{sp} \pi_2(a_2)$

Ex.: Assume π_i 1-1,

Then: $\|x\|_\beta := \|\pi_1 \otimes \pi_2 (x)\|$ is a C^*-cross-norm.

Def.: Let $x \in \mathcal{A}_1 \otimes_{alg} \mathcal{A}_2$.

$$\|x\|_{min} := \sup \left\{ \|\pi_1 \otimes \pi_2 (x)\| \;\middle|\; \pi_i : \mathcal{A}_i \to B(\mathcal{H}) \text{ unital } *\text{-homs} \right\}$$

Rmk: This is a C^*-cross-norm!

Thm: $\|\cdot\|_{min}$ is the minimal C^*-cross norm on $\mathcal{A}_1 \otimes_{alg} \mathcal{A}_2$.

Pf: Takesaki, Theo of OA, vol. 1, Thm. IV. 4.19

Cor. [12.2]:
If $\pi_i : \mathcal{A}_i \to B(\mathcal{H}_i)$ is 1-1, then

$$\|x\|_{min} = \|\pi_1 \otimes \pi_2 (x)\|$$

Pf: $\|\pi_1 \otimes \pi_2 (x)\|$ defines a C^*-cross-norm.
$\|x\|_{min} \geq \|\pi_1 \otimes \pi_2 (x)\|$ by def. Thm. $\Rightarrow$ "=". $\square$

Rmk:
1) If $\mathcal{A}_i \subseteq B(\mathcal{H}_i)$, then $\|\cdot\|_{min} = \|\cdot\|_{sp}$.

2) Let $\mathcal{A}_i \subseteq \mathcal{B}_i$, then $\|\cdot\|_{min}$ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the same as the one inherited from the inclusion

$$\mathcal{A}_1 \otimes_{alg} \mathcal{A}_2 \longrightarrow \mathcal{B}_1 \otimes_{min} \mathcal{B}_2$$

Pf: If $\pi_i : \mathcal{B}_i \to B(\mathcal{H}_i)$ is a unital 1-1 $*$-hom., it restricts to a unital 1-1 $*$-hom.

$$\pi_i|_{\mathcal{A}_i} \to B(\mathcal{H}_i).$$

By Cor. 12.2:

$$\underbrace{\|x\|_{min}}_{\mathcal{A}_1 \otimes_{min} \mathcal{A}_2} \stackrel{?}{=} \|\pi_1 \otimes \pi_2 (x)\| = \underbrace{\|x\|_{min}}_{\mathcal{B}_1 \otimes_{min} \mathcal{B}_2}$$

3) $\mathcal{A}_1 \otimes_{min} \mathcal{A}_2 \xrightarrow{*-iso} \overline{\mathcal{A}_1 \otimes_{alg} \mathcal{A}_2} \subseteq B_1 \otimes_{min} B_2$

"injectivity".

Def.: $\mathcal{M}_i \subseteq \mathcal{A}_i$ op. sp. Def. $\|\cdot\|_{min}$ on
$\mathcal{S}_i \subseteq \mathcal{A}_i$ op. sys.
$\mathcal{M}_1 \otimes_{alg} \mathcal{M}_2$ to be the restriction of
$\|\cdot\|_{min}$ on $\mathcal{A}_1 \otimes_{min} \mathcal{A}_2$.

Completions: $\mathcal{M}_1 \otimes_{min} \mathcal{M}_2$
$\mathcal{S}_1 \otimes_{min} \mathcal{S}_2$

Matrix norms induced by:
orders

$$M_n(\mathcal{M}_1 \otimes_{min} \mathcal{M}_2) \subseteq M_n(\mathcal{A}_1 \otimes_{min} \mathcal{A}_2) \subseteq M_n(B(\mathcal{H}_1) \otimes_{min} B(\mathcal{H}_2))$$

$$= M_n(B(\mathcal{H}_1 \otimes \mathcal{H}_2)) = B((\mathcal{H}_1 \otimes \mathcal{H}_2)^n)$$

Thm. [12.3]: Let $\mathcal{A}_i, B_i$ unital C^*-alg., $\mathcal{M}_i \subseteq \mathcal{A}_i$ op. sp.

$L_i : \mathcal{M}_i \longrightarrow B_i$ cb

Then: $L_1 \otimes L_2 : \mathcal{M}_1 \otimes_{alg} \mathcal{M}_2 \longrightarrow B_1 \otimes_{alg} B_2$

$a_1 \otimes a_2 \longmapsto L(a_1) \otimes L_2(a_2)$

extends to a cb map:

$L_1 \otimes_{min} L_2 : \mathcal{M}_1 \otimes_{min} \mathcal{M}_2 \longrightarrow B_1 \otimes_{min} B_2$ with $\|L_1 \otimes_{min} L_2\|_{cb} = \|L_1\|_{cb} \|L_2\|_{cb}$

Pf. Assume $B_i \subseteq B(\mathcal{H}_i)$.

Enough to show that $L_1 \otimes L_2$ cb with

$\|L_1 \otimes L_2\|_{cb} = \|L_1\|_{cb} \|L_2\|_{cb}$

Arveson's extension thm. [Thm. 7.5]

- Wittstoch's extension thm. [Thm. 8.2]:

  $L_i : M_i \to B(\mathcal{H}_i)$ extends to

  $\tilde{L}_i : A_i \to B(\mathcal{H}_i)$, such that $\|\tilde{L}_i\|_{cb} = \|L_i\|$,

  $\tilde{L}_i$ is cp

- Gen. Stinespring repr. [Thm. 8.4]:

  Stinespring dilation

  $\pi_i : A_i \to B(K_i)$

  $\underbrace{V_i, W_i}_{V_i} : H_i \to K_i$ bd $\quad \|V_i\| \|W_i\| = \|\tilde{L}_i\|_{cb} = \|L_i\|_{cb}$

  $\tilde{L}_i(a_i) = V_i^* \pi_i(a_i) W_i$

  $V_i^* \pi(a_i) V_i$

- $(V_1 \otimes V_2)^* (\pi_1 \otimes \pi_2)(a_1 \otimes a_2)(W_1 \otimes W_2)$

  $= \tilde{L}_1(a_1) \otimes \tilde{L}_2(a_2)$

- $\|L_1 \otimes L_2\|_{cb} \le \|V_1 \otimes V_2\| \|W_1 \otimes W_2\|$

  $= \|V_1\| \|V_2\| \|W_1\| \|W_2\| = \|L_1\|_{cb} \|L_2\|_{cb}$

$\|L_1 \otimes L_2\|_n = \sup \|(L_1 \otimes L_2 (a_{ij}))_{i,j}\|$

$a_{ij} = a_{ij}^1 \otimes a_{ij}^2$

$\|(a_{ij}^k)_{i,j}\| \le 1$

$\|(L_k(a_{ij}^k))_{i,j}\|_n \longrightarrow \|L_k\|_{cb}$

3) $A_1 \otimes_{min} A_2 \xrightarrow{*\text{-iso}}$

"injectivity",

Def.: $M_i \subseteq A_i$ op. s

$S_i \subseteq A_i$ op.

$M_1 \otimes_{alg} M_2$ to be

$\|\cdot\|_{min}$ on $A_1 \otimes_{min} A_2$.

Completions: $M_1 \otimes_{min} M$

$S_1 \otimes_{min} S$

Cor. [12.4]:

$x \in \mathcal{M}_1 \otimes \mathcal{M}_2$

$$\|x\|_{\min} = \sup \{ \|L_1 \otimes L_2(x)\| \mid L_i : \mathcal{M}_i \to B(\mathcal{H}_i), \ \|L_i\|_{cb} \le 1 \}$$

$\mathcal{S}_1 \otimes \mathcal{S}_2$

L_i CP

Pf.: Assume $\mathcal{M}_i \subseteq \mathcal{A}_i$. By minimality, $\|\cdot\|_{\min}$ on $\mathcal{M}_1 \otimes \mathcal{M}_2$ coincides with $\|\cdot\|$ inherited from $\mathcal{M}_1 \otimes_{alg} \mathcal{M}_2 \xhookrightarrow{incl} \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$.

Each $L_i : \mathcal{M}_i \to B(\mathcal{H}_i)$ cc induces a rep. $\mathcal{A}_i \to B(\mathcal{H}_i)$

Claim follows by minimality. $\square$

Let $\pi_1 : A \to B(\mathcal{H})$, $\pi_2 : B \to B(\mathcal{H})$ have commuting ranges:

$$\pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$$

This induces:

$$\pi : A \otimes_{alg} B \longrightarrow B(\mathcal{H})$$
$$\sum a_i \otimes b_i \longmapsto \sum \pi_1(a_i)\pi_2(b_i)$$

Conversely, given $\pi : A \otimes_{alg} B \to B(\mathcal{H})$, we obtain

$\pi_1 : A \to B(\mathcal{H})$

$\pi_1(a) := \pi(a \otimes 1)$

$\pi_2 : B \to B(\mathcal{H})$

$\pi_2(b) = \pi(1 \otimes b)$

These have comm. rngs.:

$$\pi_1(a)\pi_2(b) = \pi(a \otimes 1)\pi(1 \otimes b) = \pi((a \otimes 1)(1 \otimes b)) = \pi_2(b)\pi_1(a)$$

Def.: $x \in \mathcal{A} \otimes_{alg} B$

$\|x\|_{max} := \sup \{ \|\pi(x)\| \mid \pi : \mathcal{A} \otimes_{alg} B \to B(\mathcal{H}) \text{ rep.} \}$

Lem. ["Universal property of max norm"]

If $\pi_i : \mathcal{A}_i \to B(\mathcal{H}_i)$ rep. w/ comm. rngs.

$\pi_1 \otimes_{max} \pi_2 : \mathcal{A}_1 \otimes_{max} \mathcal{A}_2 \longrightarrow B(\mathcal{H})$

$\pi_1 \otimes_{max} \pi_2 (a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2)$

$$\begin{tikzcd} \mathcal{A}_1 \times \mathcal{A}_2 \arrow[rr] \arrow[dr, "(\pi_1,\pi_2)"'] & & \mathcal{A}_1 \otimes_{max} \mathcal{A}_2 \arrow[dl, "\pi_1 \otimes_{max} \pi_2"] \\ & B(\mathcal{H}) & \end{tikzcd}$$

$\pi : \mathcal{A} \otimes_\gamma B \longrightarrow B(\mathcal{H})$ by GNS

$\|\pi(x)\| = \|x\|_\gamma$

$\|x\|_{max} \geq \|x\|_\gamma$

Thm. [12.7, Arveson's commutant lifting thm.]

$\mathcal{H}, \mathcal{K}$ Hilb. sp., $B \subseteq B(\mathcal{K})$ unital C^*-alg., $V \in B(\mathcal{H}, \mathcal{K})$ s.t.

$B V \mathcal{H}$ norm-dense in $\mathcal{K}$

Then $\forall T \in (V^* B V)'$ $\exists!$ $T_1 \in B'$

s.t. $VT = T_1 V$

Obtain $\tilde{\pi} : \overset{B(\mathcal{H})}{\overset{\cup}{(V^*BV)'}} \longrightarrow \overset{B(\mathcal{K})}{\overset{\cup}{B'}} \cap \{VV^*\}'$

$*$-hom. onto.

Pf. of Thm. 12.8:

1) min. Stinespring repr. of Θ_1

$(\pi_1, \mathcal{K}_1, V_1)$

$\pi_1 : \mathcal{A}_1 \longrightarrow \mathcal{K}_1$

$V_1 : \mathcal{H} \longrightarrow \mathcal{K}_1$

$\Theta_1(a) = V_1^* \pi(a) V_1$

$\|\Theta(1)\| = \|V_1\|^2$

2) Let $\gamma_1 : (V_1^* \pi_1(\mathcal{A}_1) V_1)' \longrightarrow \pi_1(\mathcal{A}_1)' \cap \{V_1 V_1^*\}'$

$\mathcal{B}(\mathcal{H}) \qquad \mathcal{B}(\mathcal{K}_1)$

<u>Note:</u> $\operatorname{im}(\Theta_2) \subset (V_1^* \pi_1(\mathcal{A}_1) V_1)'$

3) well-def. map $\tilde{\Theta}_2 := \gamma_1 \circ \Theta_2 : \mathcal{A}_2 \longrightarrow \mathcal{B}(\mathcal{K}_1)$

Thm. [12.8]

$\Theta_i : \mathcal{A}_i \longrightarrow \mathcal{B} \subset_{CP} \mathcal{B}(\mathcal{H})$ with comm. rngs.

Then there exists a CP map

$\Theta_1 \otimes_{max} \Theta_2 : \mathcal{A}_1 \otimes_{max} \mathcal{A}_2 \longrightarrow \mathcal{B}$

with $\Theta_1 \otimes_{max} \Theta_2 (a_1 \otimes a_2) = \Theta_1(a_1)\Theta_2(a_2)$

<u>Q:</u> Can we take $\|\cdot\|_{max}$ to be the one inherited from $\mathcal{S}_1 \otimes_{alg} \mathcal{S}_2 \xhookrightarrow{incl} \mathcal{A}_1 \otimes_{max} \mathcal{A}_2$? No!

<u>Def.:</u> $(x_{ij})_{ij} \in M_n(\mathcal{S}_1 \otimes_{alg} \mathcal{S}_2)$

$\|(x_{ij})_{ij}\|_{max} := \sup \{ \|(\Theta_1 \otimes \Theta_2 (x_{ij}))_{ij}\| \mid \Theta_\ell : \mathcal{S}_\ell \longrightarrow \mathcal{B}(\mathcal{H})$ unital CP w/ comm. rngs. $\}$

$\rightsquigarrow$ $\tilde{\theta}_2$, π_1 have comm. rngs

$\rightsquigarrow$ — " — cp

$\rightsquigarrow$ can show that

$\theta_2(a_2) - W_1^* \tilde{\theta}_2(a_2) W_1$

$\in \ker(\gamma_1)$

$V_1^* = W_1^* P_1$

$(P_1 = (V_1 V_1^*)^{1/2})$

$\rightsquigarrow$ $V_1(\theta_2(a_2)) = \tilde{\theta}_2(a_2) V_1$

$\rightsquigarrow$ $V_1^* \pi_1(a_1) \tilde{\theta}_2(a_2) V_1 = V_1^* \pi_1(a_1) V_1 \theta_2(a_2) = \theta_1(a_1)\theta_2(a_2)$

$\rightsquigarrow$ min. Stinespring rep. for $\tilde{\theta}_2$

$\pi_2 : \mathcal{A}_2 \to \mathcal{B}(\mathcal{K}_2)$

$V_2 : \mathcal{K}_1 \to \mathcal{K}_2$

$\tilde{\theta}_2(a_2) = V_2^* \pi_2(a_2) V_2$

$\rightsquigarrow$ Let γ_{2} : $(V_2^* \pi_2(\mathcal{A}_2) V_2)' \to \pi_2(\mathcal{A}_2)' \cap \{V_2 V_2^*\}'$

$\rightsquigarrow$ $\tilde{\pi}_1 := \gamma_2 \circ \pi_1$

$\hookrightarrow$ $*$-hom

$\hookrightarrow$ $\tilde{\pi}_1(\mathcal{A}_1) \subseteq \pi_2(\mathcal{A}_2)'$

$\hookrightarrow$ $V_2^* \pi_2(a_2) \tilde{\pi}_1(a_1) V_2 = \tilde{\theta}_2(a_2) \pi_1(a_1)$

universal prop. of max tensor prod.

$\rightsquigarrow$ $\pi : \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \to \mathcal{B}(\mathcal{K}_2)$

$\pi(a_1 \otimes a_2) = \tilde{\pi}_1(a_1) \pi_2(a_2)$

$\rightsquigarrow$ $V = V_2 V_1 : \mathcal{H} \to \mathcal{K}_2$

Pf. of

1) min.

$(\pi_1$

2) Let

Note

3) well-

Pf. of Thm. 12.8:

1) min. Stinespring repr. of Θ_1

$(\pi_1, \mathcal{K}_1, V_1)$

$\pi_1 : \mathcal{A}_1 \to \mathcal{K}_1$

$V_1 : \mathcal{H} \to \mathcal{K}_1$

$\Theta_1(a) = V_1^* \pi(a) V_1$

$\|\Theta(1)\| = \|V_1\|^2$

$\Theta : \mathcal{A}_1 \otimes_{max} \mathcal{A}_2 \to \mathcal{B}(\mathcal{H})$

$\Theta(x) = V^* \pi(x) V$ CP

$\Theta(a_1 \otimes a_2) = \Theta_1(a_1)\Theta_2(a_2)$

2) Let $\gamma_1 : \underset{\mathcal{B}(\mathcal{H})}{\underset{\cap}{(V_1^* \pi_1(\mathcal{A}_1) V_1)'}} \to \underset{\mathcal{B}(\mathcal{K}_1)}{\underset{\cap}{\pi_1(\mathcal{A}_1)' \cap \{V_1 V_1^*\}'}}$

Note: $\mathrm{im}(\Theta_2) \subset (V_1^* \pi_1(\mathcal{A}_1) V_1)'$

3) well-def. map $\tilde{\Theta}_2 := \gamma_1 \circ \Theta_2 : \mathcal{A}_2 \to \mathcal{B}(\mathcal{K}_1)$

Thm. [12.8]

$\Theta_i : \mathcal{A}_i \to \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ CP with comm. rngs.

Then there exists a CP map

$\Theta_1 \otimes_{max} \Theta_2 : \mathcal{A}_1 \otimes_{max} \mathcal{A}_2 \to \mathcal{B}$

with $\Theta_1 \otimes_{max} \Theta_2 (a_1 \otimes a_2) = \Theta_1(a_1)\Theta_2(a_2)$

Q: Can we take $\|\cdot\|_{max}$ to be the one inherited from $\mathcal{S}_1 \otimes_{alg} \mathcal{S}_2 \xhookrightarrow{incl} \mathcal{A}_1 \otimes_{max} \mathcal{A}_2$? No!

Def.: $(x_{i,j})_{i,j} \in M_n(\mathcal{S}_1 \otimes_{alg} \mathcal{S}_2)$

$\|(x_{i,j})_{i,j}\|_{max} := \sup \{ \|(\Theta_1 \otimes \Theta_2(x_{i,j}))_{i,j}\| \mid \Theta_\ell : \mathcal{S}_\ell \to \mathcal{B}(\mathcal{H})$ unital CP w/ comm. rngs. $\}$