Ch B Abstract op systems.

$$I \subseteq \mathcal{A} \subseteq B(H)$$

$\mathcal{A}/I$ realized?

Choi-Effros Thm.

- Ruan Thm

Reminder

Concrete op syst.

$$\underset{\text{subsp}}{S \subseteq} \underset{\substack{\text{unital} \\ C^*}}{\mathcal{A}} \subseteq B(H)$$

$S = S^*$ $1 \in S$

ordered $S \cap \mathcal{A}_+$

R

Def $\underline{S}$ v.s over $\mathbb{C}$

$^*: S \longrightarrow S$ involution.

$$S_h := \{ s \in S \mid s = s^* \}$$

Call $*$-vector space.

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eg $a \in S$

$a \geq 0$ iff $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \geq 0$ (?)

Def. Mat. ordered $*$-VS.

S $*$-VS

assume $\forall n$ $\underline{C_n} \subseteq_{\text{wedg}} M_n(S)_h$.

- $C_n \cap (-C_n) = \{0\}$
- $\forall m$ $A \in M$ (C) of appropriate shape
  $A^* C_n A \subseteq C_m$

$(S, (\underline{C_n}))$

Ch B Abstract op systems.

S, T – $*$-vs.
mat ordered

$\phi : S \longrightarrow T$

– compl. if $\phi_n(C_n^{(S)}) \subseteq C_n^{(T)}$

compl order isomorph
ϕ^{-1} exists and is compl pos.

Def ( )(arh.) order unit

S mat ord $*$-vs.

– call $e \in S_h$ order unit if $\forall x \in S_h \; \exists r > 0$

[e, e]

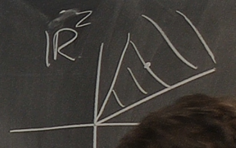
$re + x \in C_1$

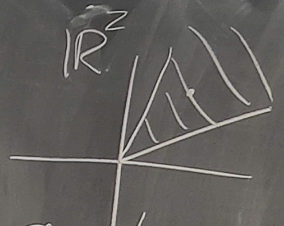
$(-re \leq x \leq re)$

e archemedean

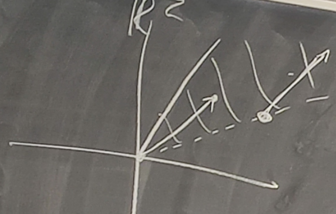
$\forall x \in S_h \; \forall r > 0$

$(re + x \in C_1 \Rightarrow x \in C_1)$





arch



not arch

∘ call e a matrix ord. unit if $\forall n$ $I_n := \begin{bmatrix} e & & \\ & \ddots & \\ & & e \end{bmatrix}$ is an orderunit

∘ <u>arch</u> mat ord unit: $\forall n$ I_n is arch order unit.

Def. <u>Mat. ordered $*$-VS</u>

S $*$-VS

assume $\forall n$ $\underline{C_n} \subseteq_{\text{wedg}} M_n(S)_h$.

∘ $C_n \cap (-C_n) = \{0\}$

∘ $\forall m$ $A \in M$ of appropriate shape (1)

$A^* C_n A \subseteq C_m$

$(S, (\underline{C_n}))$

Observations

- order units ≥ 0
- matrix o.u. $\Rightarrow$ cones gen.

$$S_h = C_1 - C_1$$
$$M(S)_h = C_n - C_n$$

- concrete op syst. $\subseteq B(H)$

- is a $*$-vs.
- mat ord.
- arch mat ord unit.

- $\forall n$ $\|\cdot\|_n$ is a norm.
- $\forall n$ $C_n \underset{\text{closed}}{\subseteq} M_n(S)$

prop 13.3.

use $A^* C_n A \subseteq C_m$

e arch mat ord unit.

Ch 13 Abstract op systems

S op system ($*$-vs, mat ord)

$s : M_n(S) \longrightarrow \mathbb{C}$

$\phi : S \longrightarrow M_n(\mathbb{C})$

$$s\left(\begin{bmatrix} 0 & 0 \\ \boxed{\ } & - \\ 0 & 0 \end{bmatrix}\right) \longrightarrow \mathbb{C}$$

ϕ_{ij}

$$\phi(s) := \begin{bmatrix} \phi_{ij}(s) \end{bmatrix}$$

Prop 13.2

$(S(C_n))$ mat ord $*$-vs

Fix $n \in \mathbb{N}$

$s : M_n(S) \longrightarrow \mathbb{C}$

$\phi : S \longrightarrow M_n(\mathbb{C})$

TFAE

i) $s(C_n) \geq 0$

ii) $\phi : S \rightarrow M_n(\mathbb{C})$ n-pos

iii) ϕ is comp pos.

Thm 13.1 Choi-Effros

S $*$-VS
mat ord
e arch mat ord unit

exists H
exists op syst $S_1 \subseteq B(H)$
exists complete ord. isomorph
$\varphi : S \longrightarrow S_1$
$\varphi(e) = Id_H$

- $\forall n$ $\|\cdot\|_n$ is a norm.
- $\forall n$ $C_n \underset{\text{closed}}{\subseteq} M_n(S)$

prop 13.3

use $A^* C_n A \subseteq C_m$

e arch mat ord unit.

$\forall n \in \mathbb{N}$.

$$P_n := \{ \phi : \underline{S} \to M_n(\mathbb{C}) \mid \phi \text{ comp pos unital } \phi(e) = Id_n \}$$

$$\underline{P} := \{ (n, \phi) \mid n \in \mathbb{N}\ \phi \in P_n \}$$

$$J : S \longrightarrow \ell^\infty \bigoplus_{(n,\phi)\in P} M_n(\mathbb{C}).$$

$$s \longmapsto (\phi(s))_{(n,\phi)\in P}$$

$$\|\phi(s)\| \leq \|s\|$$

Claim: J compl. ord iso onto $J(S)$.

Claim $[x_{ij}] \in C_n$ iff $J_n[x_{ij}] \geq 0$

$\overset{ok}{\Longrightarrow}$

$\overset{?}{\Longleftarrow}$

($\Leftarrow$) contra pos

Take $[y_{ij}] \notin C_n$ (show $J_n[y_{ij}] \not\geq 0$)

find $(n, \psi) \in P$ $\Psi_n([y_{ij}]) \not\geq 0$.

exists $S : M_n(S) \longrightarrow \mathbb{C}$ pos.

$S([y_{ij}]) < 0$.

ϕ

$[y_{ij}] \not\geq 0$

$\not\geq 0.$

$S.$

$\phi : S \longrightarrow M_n(\mathbb{C})$

$S[x_{ij}] = [\cdots] \phi_n([x_{ij}]) \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$

$\phi_n([y_{ij}]) \not\geq 0.$

$\underline{S}$ pos.

ϕ compl pos

Defin $P := \phi(\underline{e}) \in M_n(\mathbb{C})$

Two cases

P invertibl

$\exists A \quad A^* P A = Id_n$

$\psi(x) := A^* \varphi(x) A \quad x \in S$

ψ compl pos
unital

$[\cdots] \psi_n([y_{ij}]) \begin{bmatrix} A^{-1} e \\ \vdots \\ A^{-1} e_n \end{bmatrix}$ $\qquad \psi_n([y_{ij}]) \not\geq 0.$

$= S([y_{ij}]) < 0$

≥ 0

$J_n[y_{ij}] \not\geq 0$

$[y_{ij}]) \not\geq 0$

pos.

Case $\underline{P}$ not inv.

claim. $\ker P \subseteq \ker \phi(x)$ for $x \in S$, $\|x\| \leq 1$

$(\ker P)^\perp \supseteq (\ker \phi(x))^\perp$

Define Q orth proj onto $(\ker P)^\perp$:

$Q\phi(x)Q = \phi(x) \quad \forall x \in S$

$k := \operatorname{rank} Q$

Choose A $n \times k$

B $k \times n$

So that.

$A^* P A = Id_k$

$AB = Q$

$\psi(x) := A^* \varphi(x) A \quad x \in S$

compl. pos unital

$[\cdots] \psi_n[y_{ij}] \begin{bmatrix} Be_1 \\ \vdots \\ Be_1 \end{bmatrix}$

$= S[y_{ij}] < 0 \qquad J[y_{ij}] \not\geq 0$

J is compl. order iso onto $J(S)$. $\square$

Def matrix normed sp

V v.s. over $\mathbb{C}$

$M_{m\times n}(V)$

$\forall n,m$

$\|\cdot\|_{m,n}$ on $M_{mn}(V)$

$A \quad B \in M_{\dots}(\mathbb{C})$

$\|A\cdot X\cdot B\| \le \|A\|\|X\|\|B\|$

$X \in M_{mn}(V)$

Def L^∞-normed sp.

$X \in M_{mn}(V) \qquad Y \in M_{pq}(V)$

$\underline{\|X \oplus Y\|}_{(m+p),(n+q)}$

$= \left\| \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right\|$

$\overset{\text{easy}}{\geq} = \max\{\|X\|_{\dots}, \|Y\|_{\dots}\}$

Thm

Thm 134 (Ruan)

V mat normed.

TFAE:

i) V realized $\exists H$

$\exists \varphi : V \to B(H)$

φ compl. isom.

ii) V is L^∞-normed

$\psi(x) := A^* \varphi(x) A \quad x \in S$

ψ compl. pos unital

$[\ - - - \ ] \, \underline{\psi_n} [y_{ij}] \begin{bmatrix} Be_1 \\ \vdots \\ Be_1 \end{bmatrix}$

$= S[y_{ij}] < 0 \qquad J[y_{ij}] \not\geq 0$

J is compl. order iso onto $J(S)$. $\boxed{\not{A}}$

i ⇒ ii easy
ii ⇒ i)

Define $V^* := \{ v^* \mid v \in V \}$

$$S := \left\{ \begin{bmatrix} \lambda & v \\ w^* & \mu \end{bmatrix} \middle| \begin{array}{l} \lambda, \mu \in \mathbb{C} \\ v, w \in V \end{array} \right\}$$

$\cdot^* : S \longrightarrow S$

$$S_h = \left\{ \begin{bmatrix} r_1 & v \\ v^* & r_2 \end{bmatrix} \middle| \begin{array}{l} r_1, r_2 \in \mathbb{R} \\ v \in V \end{array} \right\}$$

Def

$$C_1 := \left\{ \begin{bmatrix} r_1 & v \\ v^* & r_2 \end{bmatrix} \middle| \; \|v\|^2 \leq r_1 r_2 \right\}$$

$$C_n := \left\{ \begin{bmatrix} P & X \\ X^* & Q \end{bmatrix} \middle| \begin{array}{l} P, Q \geq 0 \\ M_n(\mathbb{C}) \\ \forall \varepsilon > 0. \\ \|(P+\varepsilon)^{-\frac{1}{2}} X (Q+\varepsilon)^{-\frac{1}{2}}\| \leq 1 \end{array} \right\}$$

"Shuffle"

$$e := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Claim: $(S(C_n))$ $*$-VS
mot ord
e Arch mot ou.

Take S

Take $S \in C_n$ $\quad A \in M_{m,n}(\mathbb{C})$

$$\underline{S} = \begin{bmatrix} P & X \\ X^* & Q \end{bmatrix}$$

$$A \cdot S \cdot A^* = \begin{bmatrix} APA & AXA^* \\ AX^*A^* & AQA^* \end{bmatrix}$$

claim $ASA^* \in C_m$.

$\forall\ \varepsilon > 0$

$$\|(APA^* + \varepsilon)^{-\frac{1}{2}} AXA^* (AQA^* + \varepsilon)^{-\frac{1}{2}}\| \leq 1$$

therefore $ASA^* \in C_m$

Take $0 < \delta \leq \varepsilon / \|A^*A\|$.

$$(APA^* + \varepsilon)^{-\frac{1}{2}} A X A^* (AQA + \varepsilon)^{-\frac{1}{2}}$$

$$= \boxed{(\quad) A (P + \delta)^{+\frac{1}{2}}} =: D \leq \|\ \| \leq 1$$

$$\boxed{(P + \delta)^{-\frac{1}{2}} X (Q + \delta)^{-\frac{1}{2}}}$$

$$\boxed{(Q + \delta)^{\frac{1}{2}} (\quad)} =: E$$

Choice of δ implies

- $\underline{A(P+\delta)A^* \leq APA^* + \varepsilon}$.

$D^*D \leq I$

$\|D\| \leq 1$ Similarly $\|E\| \leq 1$

$i \Rightarrow ii$ easy
$ii \Rightarrow i$)

$i : V \to S$
$v \longmapsto \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$

claim v is compl isom.

$\underline{i_n[v_{ij}]} \simeq \begin{bmatrix} 0 & [v] \\ 0 & 0 \end{bmatrix}$

$\left[\begin{smallmatrix} [& ^{\circ}] \\ \{ & ^{\circ}] \end{smallmatrix}\right.$

$\left\| \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} \right\|$

$= \inf \left\{ r \;\middle|\; \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{rI} & 0 & V \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{rI} \\ V^* & 0 & & \end{array}\right] \in C_{2n} \right\}$

$= \inf \left\{ r \;\middle|\; \forall \varepsilon > 0 . \left\| (r+\varepsilon)^{-\frac{1}{2}} \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} (r+\varepsilon)^{-\frac{1}{2}} \right\| \leq 1 \right\}$

$= \inf \left\{ r \;\middle|\; \forall \varepsilon > 0 \; \left\| \begin{bmatrix} \cancel{0} & V \\ \cancel{0} & \cancel{0} \end{bmatrix} \right\| \leq r + \varepsilon . \right\}$

$= \inf \left\{ r \;\middle|\; \forall \varepsilon \; \| V \| \leq r + \varepsilon \right\}$

$= \| V \|_{n \times n}$ $\quad$ i compl isom. $\boxtimes$

take

A

cl

$\forall$

t